CAPITALIST CRISIS and HUMAN PROGRESS

Growth of the Productive Forces, Technodepreciation of Fixed Capital Value, Formation and Accumulation of *Capitalized* Fictitious Value, Expanded Reproduction and Consumption of 'Fictitious *Use*-Value' Emergence of a Continually Inflationary Pricing Regime, and Global Human Social Reproduction --

Preparatory Steps Toward A Dialectical-Mathematical Model of the *Historical* Self-Critique of Capital, Prior to the Deluge of Its Historical Self-Completion, by way of an Immanent Critique of Mathematical Dynamics

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III. Appendix: Basic Papers on Hypernumbers

1. C. Musès, "Applied Hypernumbers: Computational Concepts," Applied Mathematics and Computation, 3 (1977), pp. 211-226.
2. C. Musès, "Hypernumbers II: Further Concepts and Computational Applications," Appl. Math. and Comput., 4 (1978), pp. 45-66.
3. C. Musès, "Computing in the Bio-Sciences with Hypernumbers: A Survey," Internat'l Journal of Bio-Medical Computing, 10 (1979), pp. 519-525.
4. C. Musès, "Hypernumbers and Quantum Field Theory (with a Summary of Physically Applicable Hypernumber Arithmetics and their Geometries)," Appl. Math. and Comput., 6 (1980), pp. 63-94.

Reference Notes

SUMMARY

of the Mathematical Part

In the endeavor to mathematically model hitherto neglected, nonlinear dimensions of economic dynamics, a thorough examination of mathematical methods, both traditional and newly available, from a perspective treating nonlinear processes as primary, was found to be prerequisite. Among newly available mathematical tools, our examination focuses upon the hypernumbers.

Hypernumber methods have already yielded fruitful and surprising results in the domain of linear algebra. These include (1) exhibition of inverses for certain singular matrices [see Section IV, papers 1 (p. 217) and 2 (p. 53)]; (2) detection of hitherto unnoticed nonassociative cases of matrix multiplication [see Sec. IV, paper 2, p. 47]; (3) discovery of a general formula for square roots of a class of (nilpotent) matrices hitherto thought to lack meaningful or expressible fractional powers [Sec. IV, paper 4, pp. 86-87], and (4) discovery of a class of non-matrix operators having matrix-valued products [Sec. IV, paper 2, p. 49].

The present paper applies hypernumber methods to "nonlinear algebra" and "nonlinear arithmetic," meaning the calculus of nonlinear operators encountered in the domain of integrodifferential equations. The goal is to discover exact solutions -- in hypernumber closed form -- to equations of the hitherto largely intractable nonlinear type.

A central thesis of the paper is that algebra and function theory draw ultimately upon arithmetic for the phenomena they exhibit. This suggests that the "exotic" functional and algebraic properties -- including noncommutative and nondistributive behaviors -- needed for nonlinear models imply, at bottom, new kinds of numbers. In particular, they imply "nonlinear numbers" -- nondistributive, noncommutative operators grasped as (hyper)numbers.

The Fundamental Theorem of Algebra already requires extension of arithmetic to include the linear hypernumber i. Likewise, we conjecture, analytical expressibility of the functions which fulfill general nonlinear dynamical equations requires yet further extension, incorporating the recently discovered hypernumbers ε, w, p, q, m, Ω, and v.

Geometrically, this implies explicit employment of spaces beyond $\mathbb{R}^n$ -- anisotropic metaspaces with hypernumber axes -- as the state-space. $\mathbb{C}^1$, with one hypernumber axis, already arises as the natural solution-space even for certain linear dynamical systems (though these Complex-valued solution-functions are customarily interpreted back into pure $\mathbb{R}$ spaces).

Specifically, we propose a generalization of the general linear dynamical system solution-function, whose representative term is of the form

$$[Ct^k e^{at}]e^{ibt} \quad \text{or*} \quad [Ct^k e^{at}]i^{f(t)} , \qquad (1)$$

to

$$[Ct^k F(e^{at}). \ . \ . \ .]e^{ubt} \quad , \qquad (2)$$

and

$$[Ct^k F(e^{at}). \ . \ . \ .]u^{f(t)} \qquad (3)$$

as the representative terms of the general nonlinear dynamical-system solution-function, wherein u can be a hypernumber other than i.

Geometrically, with appropriate values of the constants (such as k = 0), of F, f, and of u (u = i, ε, w, p, q, or m), functions of the form (3) express limit-cycle mimicking trajectories in hypernumber space.

Problems impeding final evaluation of this conjecture are discussed.

An important concept introduced in this paper is that of a "reflexive function." This refers to functions involving the self-operation of an operator or number. A nonlinear term in a differential equation, such as $ax(t)^3 = a[x(t)[x(t)[x(t)]]]$, wherein the time-changing function-value $x(t)$ operates upon itself threefold, is a rudimentary reflexive function. But it is of the "static" kind, involving a fixed number of self-operations; a fixed degree. More important are "dynamical" reflexive functions, in which the degree of self-reflexion rises with t. The classic example of a dynamical reflexive function is the Real-valued $x(t) = e^t$, so basic to the solution of linear dynamical systems.

Both (2) and (3) above exemplify dynamical reflexive functions which are hypernumber valued. In (2) the self-operating operator is (e^{ub}), in (3), it is u. The form (2) draws upon the "exponential orbit" properties of the hypernumbers, form (3) upon their "power-orbit" behavior, with the two coinciding only in the case of i (via the identity $i \equiv e^{i\pi/2}$).

Detection of a link between the "reflexive" and "self-referential" character of the paradoxes of formal logic and set theory, and the "reflexive function" character of dynamical-system nonlinearity in mathematics proper, serves as a guiding clue on the way to our solution conjecture. The paradoxes, in appropriate versions of symbolic logic notation, take the form of logical reflexive functions. It appears that the major "insolubilia" of modern logic and of modern mathematics stem from a common conceptual root-difficulty.

The hub of the mathematics section (the sub-section titled "The Ideographic Leverage Principle") proposes an interpretation of the hypernumbers and their space, $\mathbb{H}$. According to this the hypernumber unities represent "the elements of trajectories." They represent a fundamental codification and classification of dynamical system behavior, a taxonomy of trajectory-types. Real unity, +1 (or r in hypernumber notation) presides over static ("fixed pointal") and rectilinear aspects of trajectories, i over plane-circular, ε over non-planar, skew-circular, w over elliptical, p and q over lobate aspects, and so on.

The hypernumbers thus provide, we propose, a basic "vocabulary" or "alphabet" for trajectory description. Their arithmetic and algebra express a concise "calculus of trajectories," articulating solution curves which may lack readily expressible counterparts among "vector fields" confined to $\mathbb{R}^n$. This "trajectory language" can formulate the state-space behavior of systems in an explicit analytical manner utilizing the new, anisotropic analytic geometry of $\mathbb{H}$. The ten[†] unit hypernumber symbols function, that is, as a primitive spectrum of "choreograms" enabling closed-formula expression of "choreographies" for the dances, in state-space, of general dynamical systems.

It is conjectured, but not proven, that this "choreographic vocabulary" is complete for dynamical system trajectory description. This is the conjectured generalization of the Fundamental Theorem of ordinary Algebra to encompass the "integrodifferential algebra" which includes nonlinear dynamical models.

The entire discussion develops in the context of a new, "linguistic" approach to the foundations of mathematics, which differs from both logicism and formalism as well as from other schools. This approach is employed implicitly throughout, but is not itself made the object of extended discussion herein (that task awaits other work).

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[*] [This form presupposes the principal value convention $i^t \equiv e^{i\pi t/2}$, yielding $f(t) = \frac{2}{\pi}bt$ when $u = i$, and other functions, characteristic for each kind of unit hypernumber, when $u \neq i$].

[†] [ten when r, i and σ are included, with $r \equiv$ real unity, and with σ a version of the integral operator with respect to time, such that $\sigma^{-1}(\ ) \equiv \frac{d}{dt}(\ )\cdot s(t)$, where $s(t)$ denotes the Heaviside unit step function].

MATHEMATICAL PREFACE:

Hypernumbers and Linear Differential Equations

The papers on hypernumbers published so far emphasize linear algebra applications. Applications to differential equations, linear or nonlinear, ordinary or partial, have yet to appear in the journal literature.

The hypernumbers beyond i are still sufficiently unfamiliar to make jumping directly into a consideration of their involvement in the likewise less trodden territory of nonlinear differential equations a poor context in which to make first acquaintance with them. The examples which follow show the hypernumbers in action on familiar, long subdued terrain -- that of linear dynamical systems. They provide a transition to the consideration of hypernumber involvement in the nonlinear domain.

The involvement of the first-discovered hypernumber i in the solutions of linear differential equation-systems is well-known. The immanence of another linear hypernumber, ε, the proper[*] square root of Real positive unity, is not so well-known. Yet, to the extent that all linear operators can be decomposed into operations of translation, dilation, rotation and reflection, they can be expressed, in hypernumber notation, as heterogeneous sums of various subvarieties and multiples of r, i, and ε. In these sums, r denotes the identity operator (including Real unity and the various orders of I matrices), i controls rotations, and ε controls reflections. Allowing double subscripted subvarieties[†] of ε and i, all orders of n-by-n matrices can be represented, with great gain in compactness for large n, in this hypernumber "inhomogeneous sum" form -- including, of course, the coefficient arrays associated with linear differential equations.

Many have sensed a muted discomfort emanating from traditional treatments of Complex-valued state-"vector" functions arising as solutions to linear

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①

[*] [1 is said to be an *improper* square root of itself, just as a set is said to be an improper subset of itself.]

[†] [see Section IV., Basic Papers on Hypernumbers, paper 4, pp. 87-90.]

dynamical systems. Many have noted a certain awkwardness and artificiality in the means evolved to translate such solutions back into an $\mathbb{R}^n$ state space. A certain inexplicitude surrounds dimensional aspects of the concept of unit vectors in state spaces, where different axes typically scale state-variables measured in quite heterogeneous units, quantifying incommensurable qualities of the dynamical systems being modeled.

Greater explicitude regarding state-space unit "vectors" is a key to the hypernumber treatment of linear ODEs. It opens the door to the gains in both computational efficiency and conceptual clarity that such a treatment affords.

This treatment requires the admission of hypernumber coordinate-axes, i-type and also ε-type, as dimensions of state-space on an equal footing with r(eal)-axes. It means letting subspaces of $\mathbb{H}$ ($\equiv$ the space of all hypernumbers) other than $\mathbb{R}^n$ serve as the state space. And a hypernumber state space is an anisotropic state space.

The same general solution-pattern which holds for the $\mathbb{R}^n$ solution of systems of linear ordinary homogeneous autonomous differential equations with constant coefficients, continues to hold for the hypernumber state space solution, namely

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \quad \leftrightarrow \quad \vec{x}(t) = \vec{x}(0)e^{At} + C \qquad [*]$$

The difference is that explicit hypernumber values of A are employed. Thus $\vec{x}(t)$ is also hypernumber valued.

There follow three examples of 2-equation linear dynamical systems solved by hypernumber methods.

The first exemplifies i-involvement (Complex eigenvalues), so as to establish the conventions of the hypernumber notation on otherwise familiar ground.

The second exemplifies ε-involvement (counter-Complex eigenvalues).

The third deals with simultaneous involvement of all of the three main hypernumbers encountered in the domain of linear differential equations, in the form of a nilpotent compound of ε and i in addition to r.

The procedure in each example is similar. The equation-system is given. Then a determination of the specific solution-space, involving one or more hypernumber-scalar axes and hypernumber unities as unit state-"vectors", is made. The determination is made on the basis of the specific value of A in each example. The solution is then carried out, in the space so determined, using [*]. The computational ease and naturalness of this method, and the symbolic compactness of the resulting hypernumber reflexive function, is often remarkable in these two-by-two examples, and even more so for larger systems.

The preliminary section immediately following specifies the generic properties of ε-type and i-type (for comparison) hypernumbers.

0. Specs. on ε (and i)

Generic Properties of ε-type numbers	Generic Properties of i-type numbers
(subvarieties distinguished by double subscript)	(subvarieties distinguished by double subscript)
$\varepsilon = \sqrt[2]{+1}$; $(\pm\varepsilon)^2 = +1 \equiv r$	$i = \sqrt[2]{-1} = \sqrt[4]{+1}$; $(\pm i)^2 = -1 \equiv -r$
third degree idempotence: $\varepsilon^3 = \varepsilon$	fifth degree idempotence: $i^5 = i$
multiplicative inverse = self: $\frac{1}{\varepsilon} = \frac{\varepsilon\varepsilon}{\varepsilon} = \varepsilon$; $\varepsilon^{1} = \varepsilon^{-1}$.	multiplicative inverse = additive inverse: $\frac{1}{i} = -i$; $i^{-1} = -i$.
Power-Orbit: geometrical form: circle, askew in a 4-D hypernumber space --	Power-Orbit: geometrical form: unit-radius circle in a 2-D hypernumber space --
$r \perp \varepsilon_{nm} \perp i_{nm} \perp i_{\infty}$	$i_{nm} \perp r \equiv \mathbb{C}^1$
locus-formula: $(\forall k \in \mathbb{R}^1)$ --	locus-formula: $(\forall k \in \mathbb{R}^1)$ --
$\varepsilon_{nm}^{k} = \cos^2(\frac{\pi}{2}k)\underline{r} + \sin^2(\frac{\pi}{2}k)\underline{\varepsilon}_{nm} - \frac{1}{2}\sin(\pi k)\underline{i}_{nm} + \frac{1}{2}\sin(\pi k)\underline{i}_{\infty}$	$i_{nm}^{k} = \cos(\frac{\pi}{2}k)\underline{r} + \sin(\frac{\pi}{2}k)\underline{i}_{nm}$

Generic Properties of ε (cont'd.)	Generic Properties of i (cont'd.)
Exponential Orbit: geometrical form: hyperbola in a 2-D hypernumber space -- $\varepsilon \perp r$ locus formula: ($k \in \mathbb{R}^1$) -- $e^{\varepsilon\theta k} = \cosh(\theta k)\underline{r} + \sinh(\theta k)\underline{\varepsilon}$	Exponential Orbit: geometrical form: same as for Power-Orbit locus formula: same as for Power-Orbit $e^{i\theta k} = \cos(\theta \underline{k})\underline{r} + \sin(\theta k)\underline{i} = \underline{i}^{\frac{2}{\pi}\theta k}$
Basic Operational Character: geometrical movement: reflection in the $\varepsilon \perp r$ plane through a fourth dimension (change of chirality) -- 180°, skew-circular in the 4-space of the power-orbit.	Basic Operational Character: geometrical movement: rotation in the $i \perp r$ plane (90°, circular in the plane).
algebraical type: linear operator (distributive within its own sub-varieties, and distributive and commutative both with Reals) Nonassociative in some cases.	algebraical type: linear operator (distributive within its own sub-varieties, and distributive and commutative both with Reals); Associative with other linear operators.
Matrix equivalents: (some 2-by-2 cases) $\varepsilon_{10} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$; $\varepsilon_{20} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\varepsilon_{30} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\varepsilon_{00} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \equiv r_2$	Matrix equivalents: (some 2-by-2 cases) $i \equiv i_{10} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; $i_{20} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ $i_{30} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$; $i_{00} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

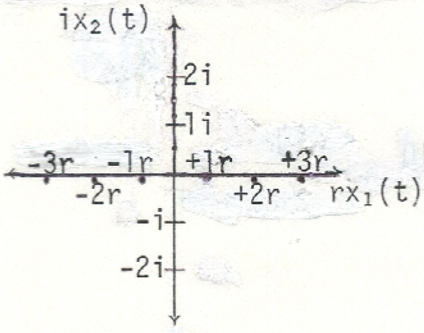
1. i-Involvement: Hypernumber Treatment

[a] the system: (p. 97, 1b, Hirsch and Smale[1974])

$$D\vec{x}(t) = \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \equiv A\vec{x}(t)$$

eigenvalues: $\lambda = \frac{-(-4) \pm \sqrt[2]{16 - 20}}{2} = 2 \pm \sqrt[2]{-1}$; $\lambda \in \{2+i, 2-i\}$

[b] determination of the state space: noting that $A = 2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 1\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = 2r + 1i_1 = 2 + i$, we designate $\mathbb{C}^1$ as the natural state space for this system, with $x_1(t)$ assigned to the r(eal)-axis, and $x_2(t)$ to the i_1-axis. This implies unit (state-)"vector" assignments as follows: $\hat{x}_1 = r = +1$; $\hat{x}_2 = i_1$.



$$D\vec{x}(t) = D\begin{bmatrix}\underline{r}x_1(t)\\ \underline{i}x_2(t)\end{bmatrix} = \frac{d}{dt}\{x_1(t)\underline{r} + x_2(t)\underline{i}\} = A\vec{x}(t) =$$

$$(2+i)\{x_1(t) + ix_2(t)\} = [2x_1(t) - x_2(t)]\underline{r} + \underline{i}[x_1(t) + 2x_2(t)]$$

$$= (2+i)\vec{x}(t).$$

[c] hypernumber reflexive function solution: Applying [*], we obtain --

$$\vec{x}(t) = x(0)e^{(2+i)t} = \vec{x}(0)e^{2t}e^{it} = [x_1(0) + ix_2(0)][e^{2t}(\cos(t)+i\cdot\sin(t)]$$

$$= \begin{bmatrix}\underline{r}[x_1(0)e^{2t}\cdot\cos(t) - x_2(0)e^{2t}\cdot\sin(t)]\\ \\ \underline{i}[x_1(0)e^{2t}\cdot\sin(t) + x_2(0)e^{2t}\cdot\cos(t)]\end{bmatrix} = \vec{x}(0)\cdot e^{2t}\cdot i^{\frac{2}{\pi}t} = \vec{x}(0)\cdot i^{\frac{2}{\pi}(1-2i)t}$$

[assuming the principal value convention that $i^t \equiv (e^{i\frac{\pi}{2}})^t \quad \forall t \in \mathbb{R}^1$].

Thus $(\forall t)\ \vec{x}(t) \in \mathbb{C}^1 \subset \mathbb{H}$.

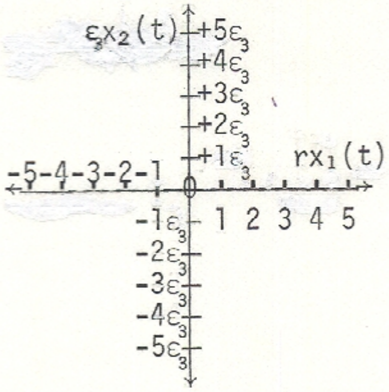
2. An Example of ε-Involvement

[a] the system: (Hirsch and Smale, ibid., 1c)

$$D\vec{x}(t) = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\cdot\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = A\vec{x}(t)$$

eigenvalues: $\lambda = \dfrac{0 \pm \sqrt[2]{+4}}{2} = \underset{\wedge}{\pm}\sqrt[2]{+1}$; $\lambda \in \{+1, -1, +\varepsilon, -\varepsilon\}$.

[b] determination of the state space: noting that, in this case, $A = \varepsilon_3$, we designate the $r \perp \varepsilon_3$ plane of $\mathbb{H}$ as the state-space for this system, since that is the plane to which the ε_3 exponential orbit is confined. We assign the first state-variable, $x_1(t)$, to the $\mathbb{R}^1$- axis, thus making r its unit state-"vector". We assign the second to the $\mathbb{E}_3$-axis, thus designating ε_3 its unit "vector". We have:


$\varepsilon_3 x_2(t)$
$+5\varepsilon_3$
$+4\varepsilon_3$
$+3\varepsilon_3$
$+2\varepsilon_3$
$+1\varepsilon_3$
$rx_1(t)$
-5 -4 -3 -2 -1
0
1 2 3 4 5
$-1\varepsilon_3$
$-2\varepsilon_3$
$-3\varepsilon_3$
$-4\varepsilon_3$
$-5\varepsilon_3$

$$D\vec{x}(t) = D\begin{bmatrix} rx_1(t) \\ \varepsilon_3 x_2(t) \end{bmatrix} = D[\ x_1(t) + \varepsilon_3 x_2(t)\ ] = A\vec{x}(t) =$$

$$\varepsilon_3[\ x_1(t) + \varepsilon_3 x_2(t)\ ] = \varepsilon_3 x_1(t) + \varepsilon_3\varepsilon_3 x_2(t) =$$

$$= \begin{bmatrix} rx_2(t) \\ \varepsilon_3 x_1(t) \end{bmatrix} \text{ (since } \varepsilon_3^2 = r), = \varepsilon_3\vec{x}(t).$$

[c] hypernumber reflexive function solution: Applying [*] yields --

$$\vec{x}(t) = \vec{x}(0)e^{\varepsilon_3 t} = [\ x_1(0) + \varepsilon_3 x_2(0)\ ]e^{\varepsilon_3 t} = [k_1 + k_2\varepsilon_3]e^{\varepsilon_3 t} =$$

$$[k_1 + \varepsilon_3 k_2][\cosh(t) + \varepsilon_3 \cdot \sinh(t)] = \underline{r}[(\frac{k_1 + k_2}{2})e^{t} + (\frac{k_1 - k_2}{2})e^{-t}] +$$

$$\underline{\varepsilon_3}[(\frac{k_1 + k_2}{2})e^{t} + (\frac{k_2 - k_1}{2})e^{-t}] = x_1(t)\underline{r} + \underline{\varepsilon_3} x_2(t) =$$

$$\begin{bmatrix} r[(\frac{k_1 + k_2}{2})e^{t} + (\frac{k_1 - k_2}{2})e^{-t}] \\ \varepsilon_3[(\frac{k_1 + k_2}{2})e^{t} + (\frac{k_2 - k_1}{2})e^{-t}] \end{bmatrix}.$$ Thus, $(\forall t)\ \vec{x}(t) \in \mathbb{R}^1 \times \mathbb{E}_3^1 \subset \mathbb{H}$.

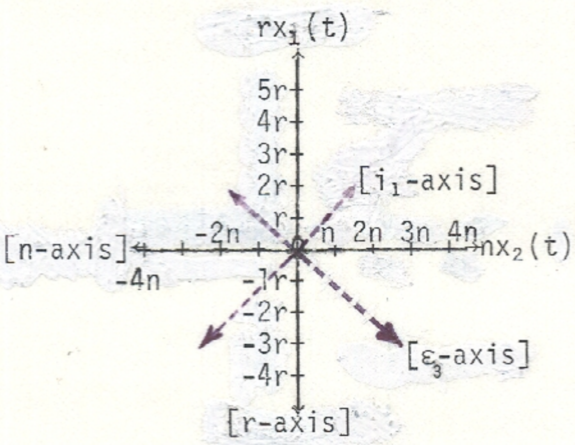
3. A Case of Compound Hypernumber ("hyperplex number") Involvement

[a] the system: (Hirsch and Smale, op. cit., p. 53)

$$D\vec{x}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\vec{x}(t) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \tfrac{1}{2}\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}\right)\vec{x}(t)$$

eigenvalues: $\lambda = \dfrac{-(-2) \pm \sqrt[2]{4-4}}{2} = 1 \pm \sqrt[2]{0}$; $\lambda \in \{1+0,\ 1-0,\ 1+n,\ 1-n\}$.

[b] determination of the state-space: Noting that here $A = r + \frac{1}{2}(i_1 + \varepsilon_3)$, we determine $(r \perp i_1) \perp \varepsilon_3$, i.e., $\mathbb{C}^1 \perp \varepsilon_3$ or $\mathbb{C}^1 \times \mathbb{E}_3$ to be the state space implicit in the system, with $x_1(t)$ assigned to the r-axis and $x_2(t)$ to the composite axis of n, where n is the nilpotent "unit" "vector" $\frac{1}{2}(i_1 + \varepsilon_3)$, i.e. $\hat{x}_1 \equiv r$ and $\hat{x}_2 \equiv \frac{1}{2}(i_1 + \varepsilon_3)$:



$$D\vec{x}(t) = (r+\tfrac{1}{2}[i_1+ \varepsilon_3])\begin{bmatrix} rx_1(t) \\ nx_2(t) \end{bmatrix} = (r+n)[x_1(t) + nx_2(t)] =$$

$$x_1(t)\underline{r} + [x_1(t) + x_2(t)]\underline{n} + \cancelto{0}{x_2(t)\underline{n}^2} = \begin{bmatrix} rx_1(t) \\ n[x_1(t)+x_2(t)] \end{bmatrix}$$

$$(\text{since } n^2 = 0)^{*}_{,} = (r+n)\vec{x}(t).$$

[c] hypernumber reflexive function solution: Applying [*] gives us --

$$x(t) = \vec{x}(0)e^{(r+n)t} = [x_1(0)\underline{r} + x_2(0)\underline{n}]e^{rt} \cdot e^{nt} =$$

$$\vec{x}(0)e^{t}[\ 1 + \frac{(nt)^1}{1!} + \cancelto{0}{\frac{(nt)^2}{2!}} + \cancelto{0}{\frac{(nt)^3}{3!}} + \ldots.] = [x_1(0) + nx_2(0)]e^{t} \cdot (r+nt) =$$

$$x_1(0)e^{t} + \underline{n}[\ x_1(0)te^{t} + x_2(0)e^{t}\ ] + \underline{n}^2[\ x_2(0)te^{t}\ ] = \{\text{since } n^2=0\}:$$

$$x_1(0)e^{t} + \underline{n}x_1(0)te^{t} + \underline{n}x_2(0)e^{t} = \begin{bmatrix} r[\ x_1(0)e^{t}] \\ n[\ x_1(0)te^{t} + x_2(0)e^{t}] \end{bmatrix}.$$

Note that the axis of the second state-variable $x_2(t)$, in this case, is not calibrated in the same unit intervals as the first. The length of one n unit is $1/\sqrt{2}$ of the r unit length: $|\hat{x}_2| = \frac{1}{\sqrt{2}} \cdot |\hat{x}_1| = \frac{|i_1|}{\sqrt{2}} = \frac{|\varepsilon_3|}{\sqrt{2}} = \frac{|r|}{\sqrt{2}}$. This is because the n-axis is not the scale of any "elemental" hypernumber scalar like i_1 or ε_3. It is, instead, the locus of all Real multiples of a compound hypernumber --

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*[n such that $n \neq 0$ but $n = \sqrt[2]{0}$ is termed a "proper" square root of zero.]

a *hypernumber complex* or "hyperplex" -- an inhomogeneous sum of two different kinds of elemental hypernumbers: $\text{n-axis} = \{z \mid z = kn = \frac{k}{2}\cdot[i_1 + \varepsilon_3], \forall k \in \mathbb{R}^1\}$[*].

4. Concluding Remarks

Hypernumber reflexive function solutions to *linear* ordinary differential equations tend to be based on the "exponential orbits[†]" of the *linear* hypernumber unities ε, i, and r, and their compounds. These solutions typically describe the "attraction" (asymptotic approach) or the "repulsion" of the system-state-point from a "fixed point" in the state space which represents a final "equilibrium" state of the system. For *nonlinear* differential equations of limit-cycle type, on the contrary, the "power-orbits[†]" of the unit hypernumbers tend to form the "analytical basis" of the solution-functions, power-orbits and exponential orbits being distinct after i. Moreover, the hypernumbers involved in the nonlinear ordinary differential equations tend to be those which are *nonlinear* operators -- i.e., which are nondistributive and noncommutative among themselves, with the Reals, and/or with other hypernumbers. These "nonlinear numbers" include w, p, q, and m.

5. References

1. Morris W. Hirsch and Stephen Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press (San Francisco: 1974).
2. (ms.) "*Economic Crisis and Technological Change*: Considerations on Modelling", Section IV. (*Appendix*: Basic Papers on Hypernumbers) *et passim.*
3. (ms.) "*Power-Orbits and Limit-Cycles*: *Hypernumber-valued Reflexive Functions as Solutions to Nonlinear IntegroDifferential Equations*", *passim.*

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[*] [see Section IV., Basic Papers on Hypernumbers, paper 2, pp. 45 to 47.]

[†] [see Section IV., paper 1, p. 213, and paper 3, *passim.*]

"...Wigner and Bochner have spoken about the miracle of miracles, the 'unreasonable effectiveness' of mathematics in the physical sciences. This miracle is contrasted by the unreasonable ineffectiveness of mathematics in the life sciences and economics."[1]

Hans Bremermann

"...The most glaring omissions have to do with the process of technical change, the role of money, and the cyclical dynamics of capitalist economies. These particular omissions are by no means accidental. It is because the treatment of these problems, which are mutually related aspects of the process of expansion of capitalist economies, constitutes the weakest link in the chain of contemporary theoretical analysis. There is, indeed, in these areas not much theory as such to be concerned with."[2]

Donald Harris

The second passage above points to some neglected dimensions of economic theory which have also been under-represented or unaccounted for entirely in mathematical models of the economic process to date. The object of this paper is to survey this uncharted hinterland of economic theory and modeling, with an eye to pinpointing the sources of these major deficiencies and, in that light, identifying emerging facilities of mathematical language that give promise of contributing to a redress of those deficiencies.

1. Some Neglected Dimensions in Economic Modeling

The deficiencies we will focus upon are those indicated by Donald Harris -- technical change, credit and monetary processes, and cyclical dynamics. As he states, they are mutually related. We will consider them principally with regard to one particular possible inter-relation. Our survey will be organized around exploration of an hypothesis concerning their interaction in producing the phenomenon of economic crisis. This hypothesis highlights the possibly seminal importance of these neglected dimensions, and the cost to the science of their neglect. They may conspire in creating some of the most important unexplained and unmastered phenomena of economic life.

We expect our investigation of the deficiencies cited in the second quotation above to yield a significant by-product bearing on the phenomenon cited in the first. We hope to remove some of the mystery surrounding the "unreasonable ineffectiveness" of mathematics in economics, and the social sciences generally, to date. The unifying theme of our explanation of this phenomenon was succinctly delineated by that consummate mathematician who, in 1945, provided the mathematical basis for the general equilibrium theory of economics[3], John von Neumann:

> "Our present analytical methods seem unsuitable for the most important problems arising in connection with nonlinear partial differential equations and in fact with virtually all types of nonlinear problems.... We are up against an important conceptual difficulty which tends to obscure the great physical and mathematical regularities that do exist..."[4]

The prerequisite to a deeper, practically more effective understanding of Nature in all the contemporary sciences is a coming to grips with the problem of nonlinearity.* The sciences of social Nature -- of that latest development of local Nature which is human society -- are no exception. Economics is no exception. As Ilya Prigogine, recent Nobel laureate for his pioneering work harnessing nonlinearity in modeling, has stated the case: "Social phenomena are described by nonlinear equations. This results directly from their relational or social character."[5] Nonlinearity would be a characteristic feature in an adequate mathematical treatment of every one of the neglected dimensions cited. Our present inability to handle nonlinearity, to solve general nonlinear equations, to explicitly formulate functions which describe nonlinear behavior, is a central cause of their neglect. "Mathematical convenience" -- designing models with considerations of mathematical tractability overriding those of descriptive accuracy -- is a major source of unrealism in prevailing economic models. And "mathematical convenience" has meant, up to the present, chiefly fictitious linearity and the "linearization" of nonlinear models.[6]

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*[See PO&LC, p. 1 ff.]

But this problem of mathematical convenience is, upon deeper reflection, not finally a matter of mathematics. As von Neumann stresses, there is a *conceptual* difficulty behind the mathematical one. In our effort to account for the "unreasonable ineffectiveness" of mathematics in economic modeling, we will have to inquire into the *non-mathematical* meaning of nonlinearity, its meaning in terms of the behavior of economic systems. We will have to discern that essential quality of economic processes which is coded into mathematical language in the form of nonlinearity. We must then pinpoint the conceptual difficulty regarding this quality which has manifested as mathematical intractability, and indicate a new theoretical concept that might resolve it. In our attempt to model the hitherto neglected economic processes cited, we will see how this concept can lead to new analytical methods promising a more effective treatment of nonlinearities.

One often detects, hovering in the background of pessimistic discussions on the problems of nonlinear mathematics, a perception of the nonlinear domain as a kind of expanse of orderless complexity, a vast morass of special cases without unifying principles.[7] The plot of the nightmare runs something like this. There is an aleph-sub-two infinite number of possible graphs in plane-space. The class of straight lines, corresponding to linear algebraic functions, is certainly much smaller, and simpler to describe, than that of the curves, corresponding to nonlinear functions. Powerful existence and uniqueness theorems assure us that nonlinear differential equations have solutions. They are not "unsolvable" in the sense that no solutions exist.[8] But if every solution to a nonlinear integro-differential equation is a new function, *sui generis*, unrelated to either the standard "elementary functions" or to other nonlinear solution-functions, then we are sunk. We would need to recognize, name, and tabulate nearly aleph-sub-two new functions. Trying to formulate solutions with these functions would be like trying to write in a language with an infinite alphabet. So runs the nightmare.

?

However, infinite multiplicity is not necessarily an obstacle to explicit representability. There are aleph-null integers on the real line. But this does not deny us the ability to name and symbolize every one of them, for they form an orderly infinitude which we have grasped. We have therefore been able to design notational facilities -- such as the infinitely extensible system of positional or place-value numerals -- for their representations. The $\mathbb{R}^2$ plane encompasses an unaccountably infinite expanse of points (vectors). Yet it takes only two "basic" vectors to "span" that entire space. Any point in that vector space can be represented as a linear combination of those two.

The chilling effect of this story on our scientific hopes depends upon the assumption of orderlessness in the nonlinear domain. But as von Neumann indicated, strong regularities are evident -- not only physical, but mathematical regularities as well -- in the domain of nonlinear phenomena. And the characteristics of the approximate versions of nonlinear solutions, expressed as trajectories in state-space, also belie this fear of structureless plethora. These solutions are highly concentrated into a very few categories of behavior, each with a very striking and highly integrated array of qualitative features.[9] In fact, the nonlinear domain shows signs of a degree of principled orderliness and theoretical interconnectedness surpassing that of the linear.[10] There may be no lack of unifying principles in the nonlinear realm. It may be simply that these principles are opaque to the old points of view developed almost exclusively in the linear domain. The new analytical methods foreseen by von Neumann may be required to reveal them.

Contemporary thinking about change in Nature moves largely within paradigms of steady states and cycles, or else of chance occurrence and random, stochastic fluctuation. Even the scientific literature exhibits many occasions of unexamined equilibristic and cyclical presuppositions. Nonlinear models reveal a new paradigm. The three major, uniquely nonlinear categories of dynamical behavior all provide examples of noncyclical but nonrandom change; of orderly aperiodicity -- including what Prigogine calls "order through fluctuation."[11] The nonlinear solution-geometry known as the "strange attractor" describes a deterministic oscillation which never repeats itself in all time, and which strongly resembles the behavior of both stochastic "noise" and random experimental error. The limit-cycle solution-geometry, despite the "cycle" in its name, describes yet another variety of aperiodic behavior. Finally, the phenomenon of "bifurcation," "branching" of solutions in nonlinear models, provides a metaphor for an even stronger notion of "aperiodicity": metamorphosis, the transition from one stage of an evolution to the next; the relatively sudden breakdown of one form and the "quantum jump" emergence into a new one after a long period of gradual change and apparent stability within the old. Many phenomena previously abandoned to linear, acausal and stochastic models, or to hypotheses of extrinsic causation, including those of quantum mechanics, may eventually find a deterministic and system-endogenous explanation in terms of nonprobabilistic but nonlinear equations.

These examples already intimate that nonlinear mathematics may contain just the metaphors we've been missing for a more effective mathematical description of crucial social and economic phenomena. In particular, the three neglected dimensions selected are interconnected by the nonlinear character they all share. But what does this commonality mean concretely in economic theory? Do these nonlinearities interact and, if so, how does this interaction look in terms of economic and social processes?

2. A Hypothesis Concerning Their Interconnection

It would be ironic, but not necessarily surprising, if the keys to accounting for major unexplained phenomena of economic life -- such as economic growth, economic crisis, and the historically observed transitions from one form of economic society to another -- were hidden under precisely those stones which economic theory had left unturned. The hypothesis investigated here suggests that such is in fact the case.

A recent paper entitled "Technological Innovation and Profit Rates" is the immediate inspiration for this hypothesis. In it, Joseph Persky and Jose Alberro of the University of Illinois Chicago Circle Campus Economics Department describe consequences of incorporating features of fixed capital into the new vintage of matrix-vector models of capitalist economies that have proliferated in the wake of works by P. Sraffa and M. Morishima. They conclude that, in an environment of rapid technical evolution: "...the extension to fixed capital suggests in a rather natural way the problems facing holders of existing capital assets, i.e. it clearly implies that they will experience capital losses (if not worse) in the short run...these capital losses can be so great as to yield a history of declining profit rates..."[12] They mention, without exploring, the possible amplification of these consequences in models encompassing credit processes, such as the operation of interest-bearing capital:

> "...our example emphasizes that technological change in a world of fixed capital engenders costs for those heavily committed to existing technologies. This almost obvious point has important implications for more complex, dynamic models. A world with a more realistic investment function including the effects of finance may be thrown topsy-turvy by such capital losses. Such short-run implications could easily be the starting point for crises in overall aggregate demand, independent of any problem of 'underconsumption.'"[13]

They also point to serious shortcomings in the writings of contemporary general equilibrium theorists with regard to these effects:

> "By focusing on a static equilibrium analysis of technological change, these writers have failed to fully describe the implications of new investment opportunities for existing capital assets. Their approach does not allow for the possibility that short-run losses may undermine the higher equilibrium profit rates expected from new technologies. Such a process might also provide a major source of periodic crises in capital accumulation since devaluation in a system which includes substantial financial leveraging is not likely to be a painless process for production."[14]

Herein is expressed the kernel of a model that links together the processes of technological change, credit, and the business cycle in an integrated explanation of economic crisis. And this theoretical conjecture is not without echoes in the more "applied" reaches of economic statistics and the empirical dialogue among business executives. In 1974, the American Manufacturers' Association Committee on Management (AMACOM) published a book entitled Return on Investment, containing the following lamentation:

> "Chart 1-1 shows that, over the past 25 years not only have profits declined in relationship to the equity of shareholders, but, much more important, the return on investment earned by durable goods manufacturers has fallen even more and now can only be classified as marginal.
>
> "In the late 1940s, business depreciated its assets over an average life of about 20 years. In recent years, however, book depreciation rates for manufacturers of durable goods now average closer to 15 years. The significance of this is very great. Economic obsolescence as a result of advancing technology has a much greater impact than it used to... While cash flows are being stimulated by the increased depreciation, the need to modernize is also demanding re-investment more quickly than before. Even with the stimulus given to cash flow by depreciation, total cash flow (profit and depreciation) in relation to investment has declined. As a consequence, business has been forced to take on additional debt. Much of this debt has been justified under the banner of expansion, but unfortunately it has in reality resulted from industry's inability to earn enough of a return to provide the capital needed to maintain itself in a viable state.
>
> "From the vantage point of shareholders, cash flow in relation to equity has improved slightly, and acted as a tranquilizer or mask to the aforementioned danger signals. But this is phoney. The 'improvement' is exclusively the result of a sharp increase in the percentage of debt as a part of capitalization. The real problem has been obscured.
>
> "We have seen that ROIs for business in total have atrophied over the years to the point of threatening the economic viability of the free enterprise system. If business, collectively, cannot generate a satisfactory return on its investment, our way of life is in deep trouble. And we are getting close to that point... the deteriorating trend of performance in U.S. industry is unmistakable. Of perhaps more significance, the absolute numbers are now becoming alarming. Who wants to put his money out to risk in U.S. durable goods manufacturers with returns like these?"

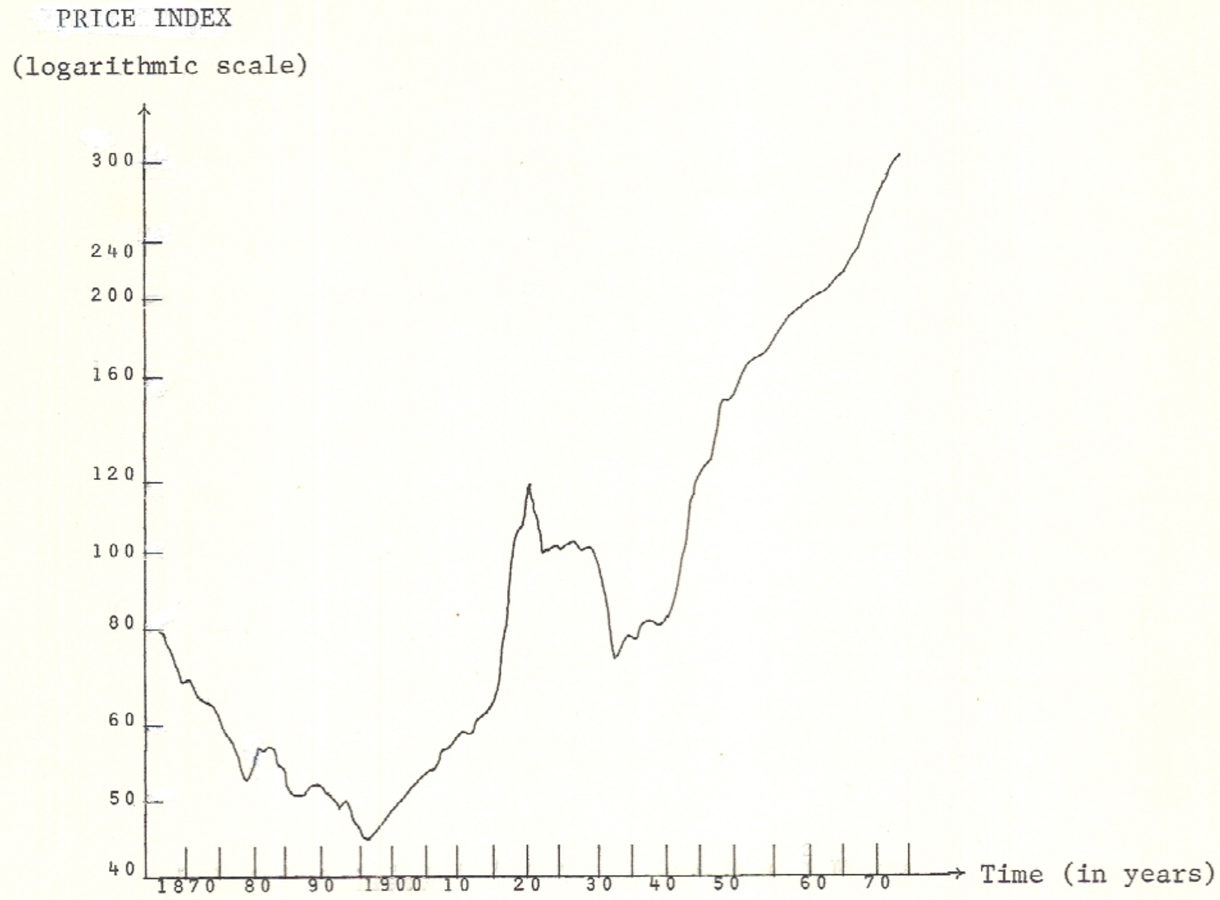
The phenomena conjectured in the first and documented in the second of our cited sources are not new in economic life. And hypotheses linking together technical advance, debt processes, and the business cycle likewise go back a long way in the history of economic theory -- at least as far as the 1904 Theory of Business Enterprise by Thorstein Veblen. In the main exposition of his version of this hypothesis, Veblen emphasizes the cheapening of capital equipment of unimproved vintages, rather than the introduction of new, technically superior vintages, as the mechanism of devaluation of previously accumulated fixed capital. It is the old prices, not the old machinery itself, which are rendered obsolete in his account. That is, Veblen concentrates on technical progress embodied in the lowered cost of production of machinery, rather than in any changes in the design of that machinery. Nonetheless, in either case, it is technical progress that is at issue. It must be kept in mind that improved design -- new vintages obsoleting old -- in the machinery used in the machine-making industries to fabricate (other) machines is a major cause of the lowered price of their output. These two aspects of devaluation of fixed capital are quite interwoven in a developing economy. With that proviso in mind, we go to Veblen:

> "The cost of production of 'capital goods' is steadily and progressively lowered, as counted in terms of the processes involved in their production. In a competitive market this is reflected, with greater or less promptitude, in the prices of such capital goods to all buyers. But the buyers whose purposes this lower scale of prices particularly subserves are chiefly the new investors who go into business in the way of new industrial establishments or extensions of the old. Each new venture or extension goes into the competitive traffic of producing and selling any line of staple goods with a differential advantage, as against those that have gone before it, in the way of a lower scale of costs. A successively smaller aggregate value of new equipment will turn out a given volume of vendible product. In so far as there is no collusive control of the output or the prices, this means that the newcomers will cut under the scale of prices at which their predecessors have been content to supply the goods. The run of competitive prices is lowered; which means that at the new competitive prices, and with their output remaining on its old footing as regards expenses of production, the older establishments and processes will no longer yield returns commensurate with the old accepted capitalization. From the inherent character of the machine industry itself, therefore, it follows that the earning-capacity of any industrial enterprise enters on a decline from the outset, and that its capitalization, based on its initial putative earning-capacity, grows progressively antiquated from the start. The efficiency of the machine process sets up a discrepancy between cost and capitalization. So that a progressive readjustment of capitalization to correspond with the lowered earning-capacity is required by the nature of the case. It is also, in the nature of the case, impracticable.

> "Insofar as the process of investment and business management involves the use of credit, in the way of interest-bearing securities or loans equivalent to such securities, this element of credit retards the readjustment by force of the fixed charges which it involves. This retardation (aided as it is by the reluctance of business men to lower their capitalization) is of sufficient effect to hinder recapitalization, on the whole, from overtaking the progressive need of it, with the result that a fair or 'ordinary' rate of profits on industrial investments is not permanently attainable in the field of open competition. In order that the rate of interest further business depression in this way, therefore, it is not necessary that the rate should rise or fall, or that it should be uniform over the field, but only that there should be a rate of interest in each case, and that there should be some appreciable volume of credit involved in industrial investments. Credit is, in fact, a ubiquitous factor in modern industrial business, and its effects in the way indicated are therefore to be counted in as a constant force in the situation.
>
> "...under conditions, such as now prevail,...the discrepancy between capitalization and earning-capacity is therefore chronic so long as no extraneous circumstances come in temporarily to set aside the trend of business affairs in this respect. It may, therefore, be said, on the basis of this view, that chronic depression, more or less pronounced, is normal to business under the fully developed regime of the machine industry."

The foregoing was penned just at the close of a protracted era in American economic history. This era was characterized by a steady deflationary trend in general price levels, including both capital goods and consumer goods levels, just as Veblen's model would suggest (see Graph I). The period also exhibited a series of severe depressions, again in agreement with Veblen's account. Indeed, the depression of 1873 was called "The Great Depression" until that of 1929 stole the name.

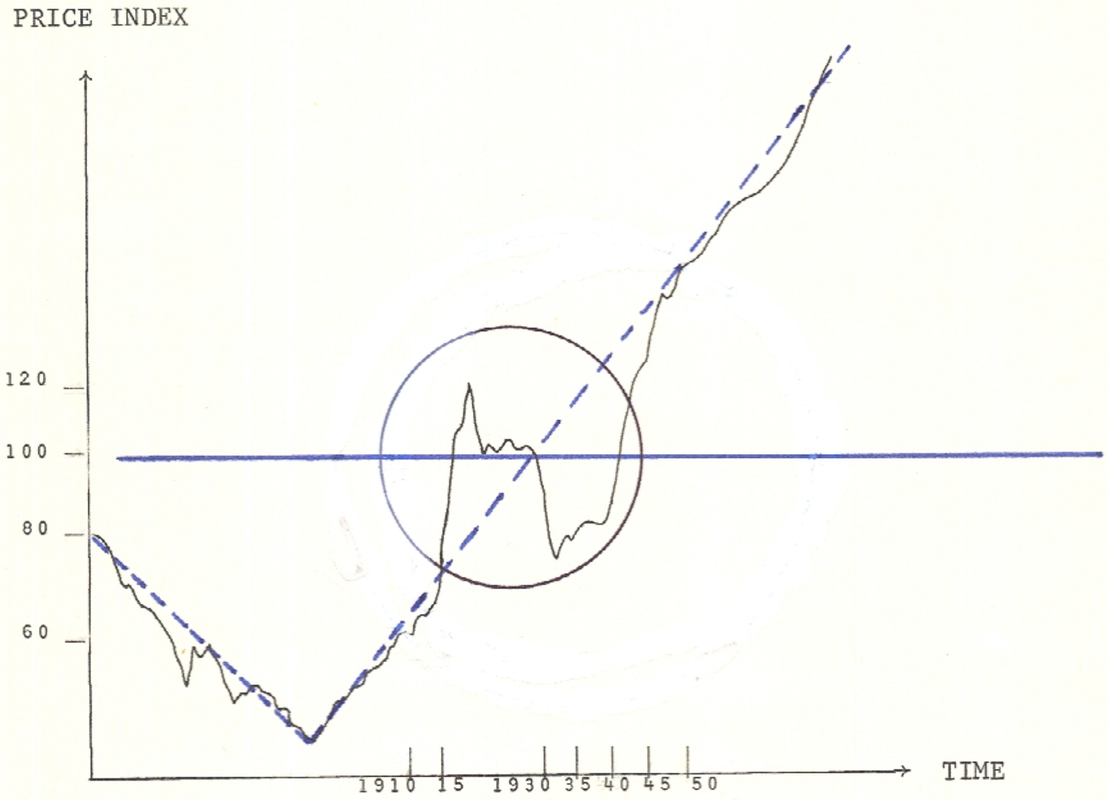
But Graph I also shows that, shortly before Veblen's book came out, a major turning point, closing the period, had been traversed. The highly articulated "V" shape of the curve highlights the sharp, qualitative distinction between the period ending and the new stage opening. The change could hardly be more dramatic: a complete about-face. The steady, prolonged fall in general price levels turned--at a point roughly coinciding with the turn of the century--into a steady, prolonged inflationary trend which persists to this day.



GRAPH I: The Price Index, 1867 to 1973 (1929 Base)

Even the interwar years, culminating in the Crash of '29, appear almost a mere "blip" in this basic inflationary trend rate. These years stand out as a highly demarcated sub-period, marked at the beginning (1913) by a faster-than-trend inflation, swiftly compensated by a two-stage deflation, bringing the sub-period to a sharply defined end (1937). The first, predominantly inflationary phase of this sub-period (1913-1929) is almost symmetrically compensated by the second, predominantly deflationary phase

(1929-1937). If you draw an "average" line through this "blip", connecting it to the end of the recovery from deflation in 1945, the resulting graph looks like an unabated continuation of the trend rate of inflation (Graph II). Note that since the vertical axis of the graph is a logarithmic scale, this roughly straight line stands for an exponential inflationary growth.



GRAPH II: Underlying Continuity of Exponential Inflationary Trend

Suppose Veblen was right about the mechanism of late 19th Century deflation: technological depreciation. Then what happened around 1900 (1896) to change all that? Surely the pace of technological evolution didn't slacken. More likely, it accelerated.[17] Yet its depreciation effects, at least upon consumer goods, did not only disappear. They reversed. How is this turn from deflation to inflation to be explained within the technological-depreciation model? Such an extreme turnabout perhaps signifies a profound institutional mutation, something like a phase transition in physical chemistry. What made deflation versus inflation the "Great Divide" between the 19th Century and 20th Century economies?

We have been able to find only one published hypothesis which uses the obsolescence-depreciation effect in its proposed explanation of 20th Century inflation. Paradoxically, the hypothesis makes this type of depreciation the central cause of that phenomenon. Technological depreciation is treated as the main cause of the general appreciation of goods-prices which defines inflation. This hypothesis also ranks as the most comprehensive, highly integrated of the obsolescence-depreciation models we have found. It weaves together technological change, credit and money processes, inflation, and business-cycle dynamics into a single pattern of explanation. It refers technological evolution to a concept of "rising general social productivity":

> What is immediately involved is the effect of rising general social productivity on the depreciation of dead capitals. By "dead capital" one can assume...the undepreciated left-over balance of machinery, equipment, and the like at the beginning of a new cycle of production...

The historical turning point from deflation to inflation is referred to a less visible process: the historical turning point in the accumulation of capital where the aggregate (average) ratio of fixed capital to circulating capital rises from less than one (before 1900) to greater than unity (thereafter):

> ...at the point at which the [ratio] of fixed capital to circulating capital reaches 1/1, a five percent increase in profit due to a five percent increase in social productivity is more or less exactly offset by a corresponding book loss on account of depreciation of the mass of fixed capital. If the ratio continues to rise, every further gain in social productivity will tend to be more than offset by depreciation losses on account of fixed capital balances...

Central to this model is a concept of "actual value" as distinguished from "fictitious value". The concept of "fictitious value" is central to this analysis of inflation:

> ...the charges for such productivity-caused depreciation of fixed capital do not involve actual value, but only fictitious (over-) valuations of fixed capital...

This concept is also utilized in explaining the paradox that apparent "overproduction" can appear as a cause of crisis because of the deterent to purchase represented by highly inflated prices, and yet reflect an actual underproduction of goods in terms of the requirements of social sustenance:

> What might appear...as a periodic excess of products is usually a reflection of a massive shortage of products (relative to human need by prevailing standards)...but under conditions of such extreme overpricing of available products that even such a shortage appears to be an excess of commodities in monetary terms...

But it is the credit and money-creation process, involving both private financial corporations and the government or public debt, which translates obsolescence-depreciation into a general appreciation of commodities, ultimately culminating in crisis:

> ...this tendency for "overproduction" of commodities does not lead immediately to depression. Through rediscounting processes always closely connected to the debt of governments, the centralized financial markets...generate credit. In the healthy aspect...capitalized credit enables sellers to dispose of surplus to buyers [who] use the surplus...(as wage commodities and means of production) to expand the scale of production of wealth... the expanded production so generated secures the value of the capitalized credit issued to set it in motion. However, the same rediscounting by central banks, or a market of financiers acting in lieu of a central bank, temporarily permits the capitalist market generally to circulate almost any amount of not only real . . . values, but also the margin of overpricing of commodities, or even the purely fictitious values of sheer speculation. Thus it would seem that, as long as credit-expansion is maintained...the monetary system provides any number of built-in stabilizers capable of postponing a crisis of "over production" indefinitely.
>
> That, however, is by no means the end of the matter. The result of what appears to be short-term monetary solutions is a general tendency for inflation.

This hypothesis uses this model of credit expansion, together with a concept of "fictitious capital", to conjecture a source for the growing illiquidity which eventuates in liquidity crisis and collapse of capital-values. "Fictitious capital" here seems to mean, roughly, a holding that accrues monetary returns by a process which does not contribute new material--"means of production" in the widest sense--to expanding social production:

> Ignoring sheer speculation and related matters for the moment, let us assume that the fictitious values added to circulation were limited to amounts needed to maintain a constant rate of profit under conditions of rising productivity...Let us call this magnitude of fictitious capitals "X"...taking the economy as a whole, X corresponds to a net excess of prices over values produced, a margin which has, in value terms, no corresponding product, wage-commodities or added means of production. Thus, although monetary expansion can readily circulate this added mass of price, no expansion of wealth results from the "investment" of X through such circulation.

By this margin of overpricing, X, is meant just that margin of depreciation of fixed capital caused by rising productivity, or technological evolution. That margin is, according to the hypothesis, transformed by the credit process into a self-augmenting or capitalized fictitious value: fictitious capital. Thus the margin of obsolescence-depreciation is proposed as the most fundamental source of crisis-precipitating illiquidity:

> The result must be, in this simplified case, that the capitalized debt created for this margin of monetary expansion could never be liquidated--that is, it could never correspond to value. It follows that the capitalized margin of debt, X, must be a self-expanding mass of potential illiquidity in the monetary system, constantly being enlarged by new production's overvaluation while the formerly created debt also multiplies through its own demands for debt-service augmentation...Credit expansion thus accomplishes the capitalization of fictitious values, so that continuation as a short-term solution to "overproduction" results in a growing potential illiquidity in the monetary system and in debt-equity relationships generally.

According to the model, this process converges upon a critical point of two-way economic instability or 'bi-instability':

> Long-term...credit expansion of this sort leads inevitably to what...economists generally describe as an accelerated shift in the "Phillips Curve", a point at which further credit-expansion to maintain "full production" must result in tendencies for rates of general inflation to spiral. At such a point, a capitalist economy faces the alternatives of proceeding through "full production" credit expansion to an inflationary explosion of Latin American dimensions, or accepting the depression.
>
> All of this...corresponds to the gross empirical phenomena of the past century's business cycles.[18]

The foregoing account of economic crisis introduces many new concepts, and raises many questions which we will have to clarify and resolve before we can construct a mathematical model to test and apply this hypothesis. Pre-eminent among these new concepts are "fictitious capital", "general social productivity", and "actual value". Pre-eminent among the questions are the following: What is the precise distinction between "actual value" and "fictitious value"? What is "general social productivity" as distinct from labor- (or other factor-) productivity defined locally for a particular enterprise unit or production process? How is a "fixed capital to circulating capital ratio" to be defined, since fixed capital is a "stock" whereas circulating capital is a "flow" category?

At first sight, the hypothesis asserted above, connecting inflation, a form of appreciation, to obsolescence depreciation, seems paradoxical. Why should the devaluation of fixed assets lead to overpricing, instead of to a lowering of prices? What prevents competition from functioning to eliminate "overpricing"?

The detailed examination of these concepts and questions belongs to Section II. We must limit this introduction to the establishment of the basic conceptual outlines of the hypothesis. But basic to the hypothesis is the notion that competition ultimately proves inadequate to enforce recognition of rising-productivity-caused depreciation of fixed assets, and to remove the margin of unrecognized obsolescence-depreciation from the prices of goods. Our provisional answers to the other questions posed above depend upon considerations not yet introduced. But a preliminary account of this notion about competition belongs here.

First of all, we have strong circumstantial evidence that, at the microeconomic level, at least for larger concerns, the "cost" of obsolescence-depreciation is routinely carried over into the price-structure, competition notwithstanding. This is so even though it is recognized that this depreciation is not a "true cost" in the "economic sense".[19] Such evidence is found, for example, in business school textbooks on management and controllership, such as the one from which the following was extracted:

> In all calculations involving the replacement or discarding of existing plant assets a distinction should be made between costs which are the result of the proposed future expenditures and those which are the result of past expenditures which cannot be recovered --or,...between "increment costs" and "sunk costs". This principle is frequently violated...(a) by adding to the investment required for a proposal the undepreciated book value of the old assets

> which are to be replaced or (b) by calculating depreciation on the old assets...by applying regular depreciation rates to their original cost, or by spreading their undepreciated book value over their estimated remaining life...As these two fallacies are surprisingly prevalent, it will be useful to understand the viewpoint of those who hold to them and sometimes defend them with almost emotional ferver. The loading of a proposed investment by adding to it the undepreciated book value of assets to be replaced is often justified on the grounds of conservatism, on the theory that the business cannot afford to make reckless investments and must recover somehow the money it spends on plant assets (underline mine); this argument is particularly likely to be advanced if the business has had some unfortunate experiences with rapid obsolescence...Actually, it is more truly conservative to acknowledge a loss...as a loss than to attempt a decision on calculations that regard the old value as still significant as an asset.[20]

The frequency of such "unfortunate experiences with rapid obsolescence", given contemporary rates of technological evolution, is acknowledged by these authorities, along with the "obsolescence", in that environment, of traditional accounting techniques treating depreciation:

> Estimating the life and residual values of assets is a necessity in the calculation of depreciation...In the past, it has been common to assume that physical factors could be estimated more certainly than economic factors. This assumption was usually carried to the point of setting up the life estimate of the asset on the basis of the physical factors only, with obsolescence treated as a special situation when it actually caused the discarding of the asset. The concept of depreciation based on only the physical factors is still the formal, acknowledged, basis of most depreciation accounting today. Actually, the pressure for shorter, "more realistic", life estimates of a great many units is an implicit recognition of the fact that relatively few assets live out their full physical lives under modern conditions.[21]

But even the proposed accounting remedy would still add "costs" of obsolescence to the price of output in the form of depreciation charges, at least whenever forecasts of technological developments proved correct:

> ...depreciation should be set up on the basis of the probable life of the unit, whatever the expected cause for its termination. If the specific assets must be scrapped before even these conservative life estimates have run, there will be the familiar adjustment of "Loss on Retirement of Plant Assets", which...is actually the loss caused by unforeseen obsolescence.[22]

Of course, price competition from firms less prudent, or from newcomers less heavily committed to the older vintages of capital equipment (or to the older prices of such equipment) could force abandonment of these higher depreciation charges. This observation leads to what may be a crux of the "fictitious capital" hypothesis: the same development which fosters fictitious capital blocks new-entry competition.

The statement of that hypothesis reproduced above stipulates that the transition to the "permanent inflation" state of the economy presupposes a preponderance of fixed to circulating capital in each production cycle, an aggregate F/C (fixed to circulating capital value ratio)$\geq$ 1. Prior to that stage in the accumulation of capital, and evolution of technology, "permanent deflation" should predominate, with a secular deterioration in the average return on investment, visible on a firm-by-firm basis. This deflation and deterioration would owe to the same causes specified in Veblen's model. But according to this hypothesis, after the turning point in accumulation marked by F/C=1, the deflationary trend inverts. Deteriorating ROI is no longer reliably visible in the individual firm. It is masked at the microeconomic level. It is transformed by credit-processes. It shows up instead at the macroeconomic level, disguised as inflation.

The connection is as follows. The high values of F/C in the leading firms make them exceedingly vulnerable to technological depreciation (of both kinds*). Potentially, they have more to lose than to gain from further technological evolution. For their gains from heightened technological productivity come via the decreased cost-composition (increased profit-composition) of their output, C. Their losses come through devaluation and discardment of portions of their capital plant, F. And for these firms, their F already outweigh their C.

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*[See p. 8 above]

However, this potential actualizes only if capital devaluation losses are enforced by new entrants to their market. New entrants, unlike the established competition, partake in the advantages of technological improvement[†] without these financial losses. The same condition which heightens potential vulnerability impedes the actualization of this potential. At least, this condition impedes actualization as losses which would register microeconomically, that is, locally to the individual firm. High F/C impedes competition. High fixed-capital intensity is a barrier to new entry. It raises the 'entry threshold', the 'quantum' of capital necessary to enter the industry and market in question. In the industries with highest F/C, such as steel, major new entry, at least domestically, becomes virtually impossible, even when their capital plant has become overwhelmingly antiquated. So the feedback of new entry competition wanes in the same movement that increases 'exposure' to obsolescence. Analyzing the hypothesis only localistically, microeconomically, we might conclude "So what? What adverse effect could a single firm charging depreciation for technically obsolete assets at their original cost have on the whole economy? Isn't the obsolescence of such assets only a technicality if not brought to bear by competing firms? How would this cause inflation? If this is overpricing--overpricing relative to what?

Milne and Laight, in a book on the economics of inland transport in the U.K., provide a glimpse of how the hypothesized inflationary process might work. They first note the differing impact of obsolescence on capital goods financed via "risk capital" (capital equity stock) versus bank loan capital.

[†] [The dimensions of such improvement include: lowered cost of acquisition, accelerated output or throughput of product, increased reliability (lowered repair/maintenance costs) and longevity, more economical use of fuel or other energy (lowered operating costs), and improved health and safety features.]

In the first case, stockholders absorb the losses. In the second, losses cannot be written off at the expense of the creditors. The loan contract must be fulfilled, or the firm can be received into bankruptcy. The principle and interest must be paid off, even though the equipment be scrapped, no longer present to produce the output, the revenue, and the profit out of which the interest was to have been covered. The purchase price of the discarded equipment is no longer a cost "in the economic sense". Replacement cost is zero, since the equipment will not be replaced. Yet that price remains a capital cost which the firm must meet out of other production. Milne and Laight refer to this obsoleted portion of capital as "redundant (financial) capital". They discuss conditions under which it may be "kept alive" by the charging of "higher-than-cost-prices":

> A firm operating in conditions of competition may have little choice...because of the likelihood that other, competing firms which have taken advantage of the technical improvement will undersell. But if the firm possesses a high degree of monopoly it can, if it is so disposed, try to recoup the loss on the old equipment out of revenue earned by the new, in which case the benefit conferred by the technical superiority of the new equipment is not being passed on to the community.[23]

This would appear to reduce the "fictitious capital" model to a simple "monopoly" model of "administered prices":

> A policy which maintains prices in excess of costs for the purpose of preventing a financial loss on past investments is indistinguishable in effect from a policy which maintains prices in excess of costs for the purpose of making "inappropriately high" profits.[24]

However, such a reduction appears to lose the essense of that hypothesis. It forfeits the explanation of the inflationary tendency.

What we have seen in the last few extracted passages is a series of descriptions which look at the process of obsolescence depreciation from the vantage point of a single enterprise. Apparently, the problem upon which the "fictitious capital" hypothesis focuses cannot be clearly perceived for an analysis which looks at the economy a single firm at a time. A crucial feature of the hypothesis is the credit process, which *globalizes* the locally-apparent effects of technological change (emphasized by the Veblen and Milne/Laight models). In the "fictitious capital" model, this process creates a trend throughout the *whole* economy not discernible in a simple sum or collection of the trends calculated for its parts, the individual firms, if these are analyzed as if in isolation from the whole process. This kind of discrepancy between a whole system analysis and a parts-taken-separately analysis is a hallmark of nonlinear systems.[25]

Of course, even in the fictitious capital hypothesis, the tendency to monopoly, engendered by rising F/C--the waning of direct, new entry competitive pressure--plays its part. But in that model, waning competitivity is not the most important consequence of rising F/C.

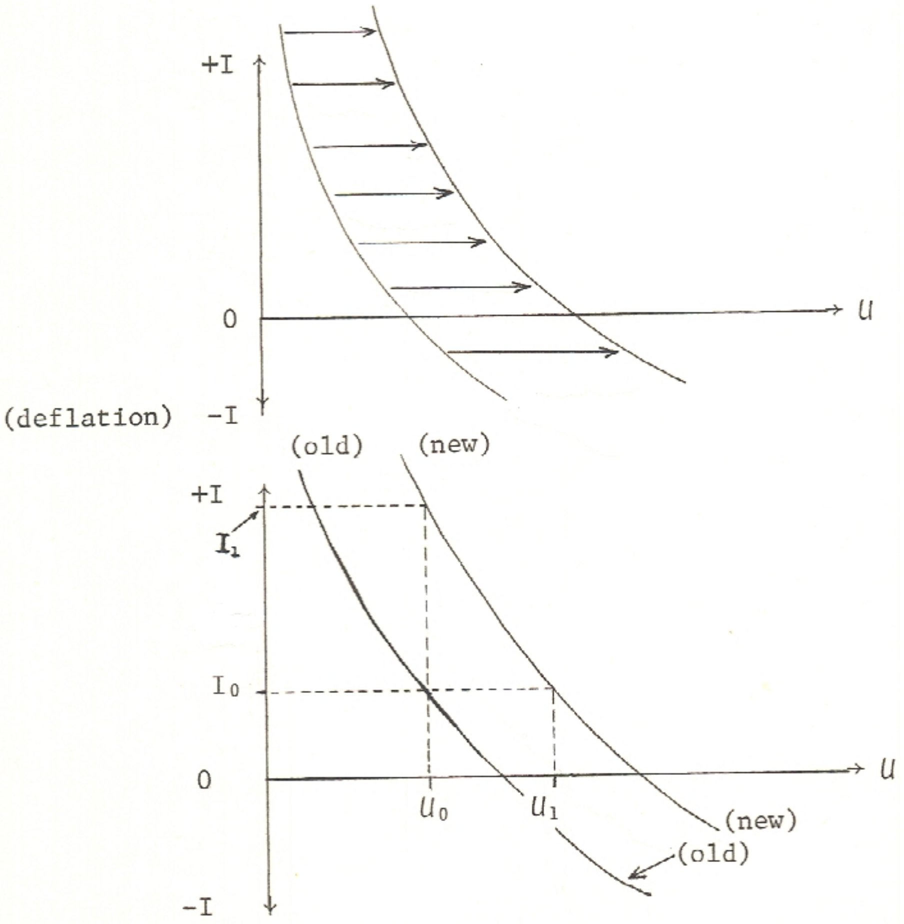
The monopoly interpretation of the last quote suggests a mere *transfer* of purchasing power or enforceable claims on production to the monopoly firm from other sectors of the economy. The fictive capital model differs decisively here. It does not imply just a relative imbalance in the *distribution* of demand-power across the expanse of the economy, while the relation between the total claims on production and total production remains roughly balanced. It implies an *absolute overage of means of demand*, at the level of the whole economy, relative to the *means of supply* or current output at prevailing prices. There is more paper right-to-demand "in circulation" than there are hard goods coming out with which to cancel that right-to-demand. This absolute overage thus manifests as inflation.

The excess or "fictive" claims on production, while arising in productive industry, become canalized and concentrated in the form of instruments of long term debt, bank loan contracts and the like. The fundamental variety of inflation in this model is 'debt-pull inflation'. Moreover, this fictive margin of claims is modeled as self-augmenting, due to the capitalization of debt. The discrepancy between current claims, and the price equivalents of current output goes on growing in this model until a critical point of illiquidity is reached, triggering the crisis collapse. This crisis can precipitate in a number of ways. These ways include (i) international monetary crisis, (ii) chain reaction bankruptcies due to a cascading sequence of broken payment committments, and (iii) sudden massive waves of layoffs as available liquidity is diverted from wages to debt service payments.

As stipulated in the quoted statement of this hypothesis, the State Debt enters decisively here. Deficit spending means that, in effect, the government prints new legal tender, lets it out to banks through the central bank or Federal Reserve, and borrows it back at interest, thereby expanding the Public Debt. This kind of deficit financing and growing National Debt is central to the more extreme stages of the model fictive capital process.

Hovering in the background of this whole discussion is a conception involving two distinct kinds of value. This conception is implicitly invoked wherever the notion of fictive value enters. There then becomes operative in the model a duality between monetized, represented, paper "demand-power" (rights and claims) and the "supply side" of value, the goods and services that contribute to further social life and production. In later sections, it will be chiefly through developing this concept of value operative in the fictive capital hypothesis, that we will clarify, in preparation for mathematical modeling, the concepts and questions raised by this hypothesis and enumerated earlier.

We close this introduction with a proposed empirical test of this hypothesis.[26] This test involves a generalization of the "Phillips Curve", or rather, of that variant of it which crosses the inflation rate with the rate of unemployment. The expectation of stability through time for the continuum of equilibria forming this "Phillips Curve" involves an assumption. This is that the demand represented by wages is the only significant ultimate or "final" demand in the economy. Otherwise, if there were another source of demand, independent of wages-demand, it might grow independently of any change in the (un)employment rate. That would drive up the inflation rate. The "equilibria" along the Phillips curve would cease to be equilibria. They would become unstable. In effect, the whole curve (all of the equilibrium points) would appear to shift to the right (see Graph III).



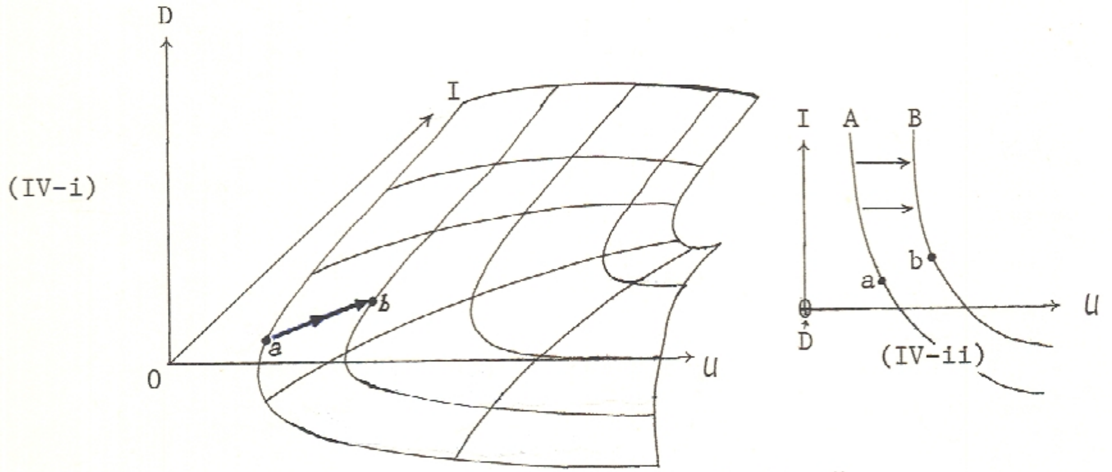
Graph III. Shift of the Phillips Curve

Holding to a given rate of inflation I_o would then require a much higher unemployment rate, U_1. Alternatively, holding to the old unemployment rate U_o would require a drastic increase in the inflation rate, from I_o to I_1.

This kind of shift has in fact been observed during recent years.[27] This phenomenon, branded "stagflation", is quite paradoxical from the traditional standpoint. The phenomenon of _simultaneous_ rises in unemployment rates and inflation rates has continued to puzzle economic theorists, and no proposed explanation has as yet been found satisfactory.[28]

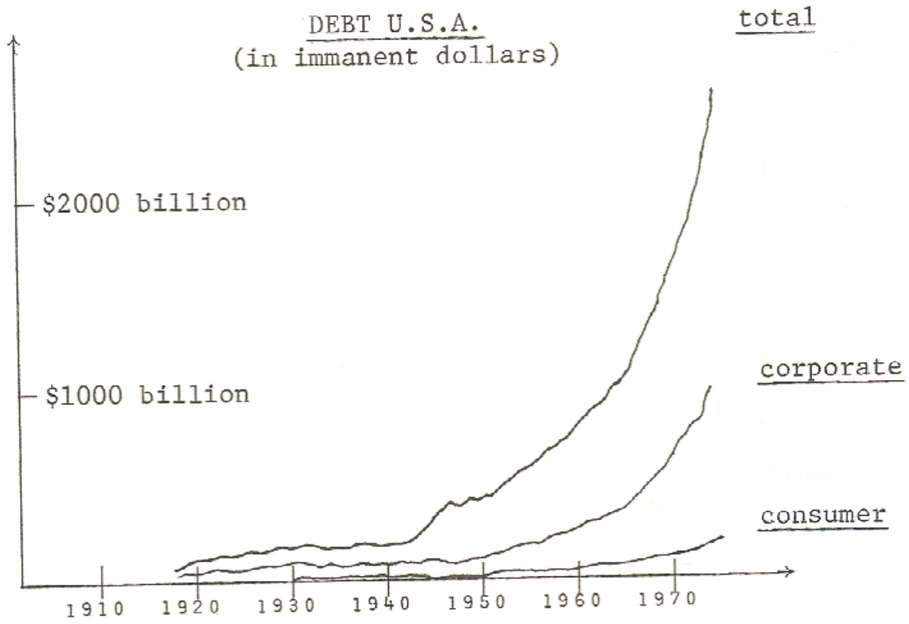
However, suppose the hypothesized 'debt-pull' demand of the margin "X" of fictitious capital were incorporated into the Phillips picture. As a dimension of demand independent of wages, we might depict it as a third orthogonal scale, measuring the ratio of fictitious capitals to total circulating capitals, X/C. A convenient estimator of this ratio in terms of available statistics might be the ratio of total public and private debt to total corporate equity, call it D.

In that case, the Phillips curve generalizes to become the "Phillips _surface_":



Graph IV. The "Phillips Surface"

A shift in the state of the economy from that combination of unemployment and inflation rates expressed by point a to that represented by b in Graph (IV-ii) would be revealed as merely a projection into the $I \perp U$ plane of a process transcending this two-dimensional description. That third-dimensional process would be the "climb" of the operating point of the economy up the side of the mound-shaped Phillips surface. This "climb" would owe to an expansion of the (self-expanding) margin of fictitious value in total circulation. That is, its direct cause would be a rise in the ratio of fictitious to total circulating capital, independent of any increase in the rate of employment. Thus this model could render the observed shifts of recent years intelligible. In the sequel, we will consider further the prospect of such a test. Here we note in passing the exponential form of the curve of total U.S. debt (see Graph V).



Graph V. U.S. Debt (in dollars unadjusted for inflation)

II. PREPARATORY CONSIDERATIONS

1. Considerations on Mathematics as a Tool of Description

Our general concern is to pinpoint certain deficiencies in current economic modeling and to identify avenues for their redress. Pursuant to this, our purview cannot stop at that side of the problem which involves only certain undeveloped aspects of economic theory. We need also consider the serviceability of the mathematical language in which, for modeling purposes, economic theory, developed or not, must find expression. We have already seen how the constraint of "mathematical convenience"--especially the constraint of linearity--has hindered the development of more realistic economic models. The undeveloped aspects of economic theory and the inadequacies of conventional analytical techniques vis-á-vis nonlinearity interact and mutually reinforce in the formation of the deficiencies we have cited.

In this subsection we address the questions "Why does mathematical convenience make for bad economic modeling?" and "What is lacking in conventional mathematics for describing nonlinear processes?" In particular, we define the fundamental nature of the nonlinearities involved in economic processes--the economic meaning of nonlinearity. This brings out a central thesis of the paper: that the mathematical problem of nonlinearity and the economic-theoretical problem of our three "neglected dimensions" are not different problems, but only different symptoms of a single underlying conceptual difficulty.

In these efforts, we will enlist a new approach, what we call the linguistic approach, to the foundations of mathematics. This approach has roots in the work of Benjamin Whorf. Whorf noted the (usually largely unconscious) ontological committments embedded in the deep grammatical structures of languages. He promulgated a "principle of linguistic relativity" according to which the natural language base of a culture group strongly influences its "natural logic", or "common sense" everyday outlook, as well as its more formal philosophical, scientific, logical and mathematical productions. In particular, Whorf traced the ontological influences of later Indo-European natural language grammar-structures into the "artificial languages" of modern symbolic logic and mathematics. He hypothesized that natural languages differing significantly in their grammatical cores would tend to nurture very different scientific paradigms, and dramatically contrasting logical and mathematical notations and methods.

Our method in this subsection is to approach mathematics as a form of written language. This language is ideographic rather than phonetic--expressed in "ideograms" rather than "phonograms". We will inspect the "ideogrammar" of this ideography to detect its hidden ontological biases. The unconscious assumptions thus uncovered turn out to be the precise source of the difficulties with nonlinearity. This method also discloses a connection between nonlinearity in mathematics and an unresolved problem of longstanding in logic--that of the semantic and set-theoretical paradoxes.

This connection, in turn, allows us to decipher the meaning, in terms of processes in the external world, of mathematical nonlinearity, i.e., to delineate the external-world phenomena to which this mathematical phenomenon refers. By this means, finally, that reform of mathematical language and "grammar" required to facilitate the treatment of nonlinearity, and the associated conceptual innovations, are identified. These reforms and innovations are then located in terms of an already existing historical tendency of longstanding within mathematical notations and conceptions, which has recently achieved a dramatic and remarkable fruition. This fruition will, in the next section, provide the most promising tools for redressing the cited economic modeling deficiencies.

To get underneath a problem as knotty and deeply-rooted as that of nonlinearity, we will have to delve deep. Perhaps paradoxically, this will at first present itself as a naive and primitive simplicity. But please bear with us through this opening gambit, serving as a necessary reconstruction of our appreciation of mathematical symbolism, and highlighting the differences between our "linguistic", and more traditional methods of mathematical exposition.

The Concept of a Process Algebra

The opacity of the symbolism in which mathematical models are expressed, to the mathematical novice, however much he may know of the concrete processes thus described, is notorious. In part, this is as much to be expected with mathematics as with the phonetic scripts of foreign languages. Beyond reach of that analogy, however, is the consideration that mathematical "script" involves a different principle of symbolization, foreign to *phonetic* scripts, native and foreign alike. This is the *ideographic* principle. It distinguishes mathematical symbols from both the *phonogramic* symbols of literal and musical notation, etc., and the *pictographic* symbols of architectural drawings, electronic circuit schematics, choreographic diagrams, roadway signs, and so on.

Mathematical symbols are chiefly ideograms. Such symbols take, broadly speaking, ideas, precise conceptions, for their values, rather than either sounds, as with phonograms, or caricatured visual images, as with pictograms.

Yet even with all this said, an aspect of the opacity remains unaccounted for. Something of this residual is captured in the query "Why does mathematics describe *processes* by *equations*?" This query is often heard from young students on first exposure to mathematical models. It is more often squelched than adequately answered. Actually there are very good answers to this question which yield important insights both about mathematics and about the concept of a dynamical law. But let us sidestep this issue for now. Suppose we try, instead, a linguistic experiment. Suppose, retaining the ideographic principle, we naively attempt to construct the least opaque, most straightforward approach imaginable to the ideographic description of processes.

One approach would be as follows. Let our symbols themselves represent processes. Let us assume we have somehow arrived at a basic spectrum of ideograms, each standing for a fundamental category of process. This spectrum would thus constitute in toto a basic "vocabulary" or alphabet of process. Suppose further that the "ideogrammar" of this "process-ideography" centered upon the following simple grammatical and syntactical principle. The interaction of processes is described by the juxtaposition of the symbols representing them, together with an arrow pointing to a symbol representing the product or outcome of that interaction:

$$ab \rightarrow c \qquad (1)$$

The use of the arrow is reminiscent of chemical "equations". It connotes "becoming", whereas the equals sign of the traditional mathematical equation formally signifies only pure, "timeless" synonymy or substitutability. Putting aside the problem of the fundamental taxonomy of process and the "alphabet of process" presupposed, there is evident in this principle a certain elegant beauty, an economy of expression and an ease of intelligibility, contrasting sharply with the opacity cited above. In fact, this principle was not pulled out of a hat. It distills the central tendency in the entire historical evolution of mathematical notation. It represents the pushing to completion of a tendency in mathematical grammar that has been building for going on two centuries now, as we shall see.

But reflection quickly points up difficulties and paradoxes for such a notation, even supposing the ready availability of a "process-alphabet". For example, a symbol by itself, without juxtaposition with anything else

$$a$$

already denotes a "process". Why shouldn't we expect such a process to "spontaneously" change into something else, a different process, after a time, even without interaction with any other.

$$a \rightarrow e$$

A process, an activity, an ongoing change--how can we expect such a thing to stay the same? But incorporating such spontaneous transitions, we lose what would seem to be a primary requirement for any workable system of notation. That requirement is stability of denotation. Without it, how would we know, in any given expression, whether a "still" stood for a, or had turned into e, or even some later term in a whole sequence of spontaneous transitions?

$$a \rightarrow e \rightarrow g \rightarrow k$$

A more devastating indictment of a symbolic system is difficult to imagine. Yet it turns out that this and related objections can be met in one stroke, by a single reform, repairing what might at first appear to be just one more of the many faults discernible in this attempt at a "process notation". The fault in question is the inexplicitude in regard to the timing of the transformations indicated by juxtaposition. The arrow gives a vague suggestion of temporality. But, for instance, how long does it take for "a-interaction-b" to yield c? The form (1) gives not the slightest clue, nor does it, as it stands, provide any natural notational facilities for the adjunction of such information.

The Noun-Verb Dichotomy

We will return to the details of this rectification, as also to the question of the process-alphabet (or process-arithmetic as we prefer to call it, for reasons which will become apparent). But for now, let us turn to explore a third objection to the form (1). For this latter objection backfires in a profoundly revealing manner. That for which it faults form (1) turns out to be that form's essential strength and advantage. The reason this objection prefers traditional mathematical notation turns out to be the essential weakness of that notation, the source of its ineffectiveness with respect to nonlinearity.

The objection emerges most clearly via analogizing (1) to the grammatical form of our natural language, as follows. Suppose we construe 'ab → c' in the form of an English sentence. We get:

a [interacting with] b [produces] c

for example:

humanity (interacting with) nature (produces) artifacts.

The trouble is, 'ab → c' in these terms is grammatically incomplete. There are "parts of speech" missing, specifically, the verbs. The verbs have to be "dubbed in" extraneously. We naturally take a, b, and c to be the nouns--a the subject, b the co-subject, or b and c both the objects. Then all the verbs are missing. How do we tell the action of the "sentence"?

In classical arithmetical notation, the institution of an exclusive class of ideograms representing "pure" operations handles this problem: +, -, x, ÷, $\sqrt{}$. The "mathematical verbs" are supplied by interjecting these symbols between the numerals, or the algebraic variables representing numerals, just as we habitually put verbs between the subject- and object-nouns in our sentences.

$$a + b \qquad a - b$$

$$a \times b \qquad a \div b$$

$$a^b \qquad \sqrt[b]{a}$$

In conventional arithmetic, this ephemeral class of "pure" operation-signs is sharply and strictly distinguished from those representing number, quantity, quantitative "substance"--the mathematical nouns[*]: 2, 3, 4, x, y, z,... . The noun-verb dichotomy of our natural language grammar is mirrored in the quantity-versus-operation division of our ideographic mathematical system. These dualities express a deepseated and tendentious ontological classification as well, tending to divide the world-view into inert, motionless, solid "substance" or "matter", and formless, disembodied modes of pure action, abstract "energy". As Whorf stated the case:

> The Indo-European languages and many others give great prominence to a type of sentence having two parts, each part built around a class of word--substantives and verbs--which those languages treat differently in grammar. As I showed in the April 1940 Review, this distinction is not drawn from nature...The Greeks, especially Aristotle, built up this contrast and made it a law of reason. Since then, the contrast has been stated in logic in many different ways: subject and predicate, actor and action, things and relations between things, objects and their attributes, quantities and operations. And, pursuant again to grammar, the notion became ingrained that one of these classes of entities can exist in its own right but that the verb class cannot exist without an entity of the other class, the "thing" class, as a peg to hang on... This contrast crops out in mathematics as two kinds of symbols --the kind like 1, 2, 3, x, y, z and the kind like +, −, ÷, $\sqrt{\ }$, log-, though in view of 0, 1/2, 3/4, π, and others, perhaps no strict two-group classification holds. The two-group notion, however, is always present at the back of the thinking, although often not overtly expressed.[29]

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[*] [See PO & LC, pp. 13 ff., The Operator Concept]

The 'Operator' Principle: Noun-Verb Synthesis

"All the verbs are missing". But here recall our opening proviso. Our symbols stand for processes. They already represent actions, activities, events. They already are "verbs", _ideographic_ verbs. Yet they also, simultaneously, perform a "noun" function. They represent processes _as substantives_, as entities, things, objects. Their "substance" _is_ a definite _kind_ of _activity_. As nouns, they are _their own_ verbs, as verbs, their own nouns. They represent, if you will, events and entities as one: "_eventities_".

In order to know or indicate the content and outcome of a symbolized interaction, there need not be interjected a separate, extrinsic and extractable, sign of pure operation between every pair of process-symbols. All the information about the interaction is already contained in the juxtaposed pair themselves. The outcome is prescribed by the meaning of the symbol on the left and of the one on the right. It is given in the natures of the two processes interacting. The "verb" or "operation" descriptive of their meeting is inherent in their natures. Any separate sign of operation would be redundant.

The form (1), thus embodies a grammatical principle of the _fusion_ of "noun" and "verb" aspects. It corresponds to a longstanding trend in the history of mathematical notation: the development of the "_operator_" conception of mathematical symbols. An operator is both a "value" and an operation sign. It can symbolize the application of an operation to another symbol-value, and it can also be the recipient of operational action performed upon it by other "operators". In the notations of ordinary algebra, matrix algebra, function theory, and in the use of the differential and integral signs in calculus, this principle is already coming to the fore.

ordinary algebra	function-notation	matrix-vector notation	differential and integral notation
$xy = z$	$f(x) = y$	$AB = C$	$Df(x) = y'$
$abbbcx = y$	$fx = y$[*]	$A\vec{x} = \vec{b}$ [†]	$\int f(x)dx$
$ab^3cx = y$			$\nabla^2\psi = 0$

But even ordinary arithmetic processes can be reinterpreted in this light with gain of insight. Ordinary numbers are also "operators" or "functions". Thus

$$2\cdot 3 = 2(3) = 6$$

Ordinary, cardinal "multiplication" is a counting process. What ordinary numbers do to one another in juxtaposition is count one another. An ordinary numeral stands for an operation of counting, and its content or 'substance' is a 'count', the result of an operation of counting already performed. The function-notation reading of expressions like the above actually clarifies this: "two *of* three equals six", "two threes equal six"[**].

Habitual, exclusive familiarity with this kind of operator--the so-called "real number" kind--actually hinders perception of the full meaning, generality, and power of the principle of juxtaposition. We need computational experience with *other kinds* of operators, which do other than count the things upon which they operate, before we can appreciate the fullness of this idea.

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[*] [See Cajori, *A History of Mathematical Notations*, vol. 2, pp. 138-9 (paragraph 105, statement attributed to Peano): "In the function f(x), the parenthesis is useless; it would be well to return to the Lagrangian fx." (See also p. 268, paragraph 643): "It is to be observed that neither Johann Bernoulli nor his successors Clairaut and D'Alembert in their earlier writings enclose the variable or argument in parentheses. ...(About the same time) Clairaut designates a function of x by πx, Φx, or Δx, without using parentheses...". (p. 272, paragraph 650): "Gauss expressed the gamma function by πz...".]

[†] [Vectors appear an exception to operatorial principles. They have several different "products", thus need several different "pure operation" signs -- "dot" and "cross", etc. -- intervening between interacting vectors' symbols to distinguish the different kinds of multiplicative interaction. We will not take up this issue here. However, hints of a comprehensive hypernumber approach to vectors as points in $\mathbb{R}^n$ emerge at intervals throughout the sequel. Banesh Hoffman, in *About Vectors*, gives clues to such an approach, pointing out the dimensional anomalies of the "dot"(p. 128) and especially the "cross" products(pp. 105-6;123); the quaternion-valued character of vector division(pp. 106-10), and by his *tensor* approach to vector processes.]

[**] [see PO&LC, p. 15.]

It is important to note that the generalized operator principle is not an arbitrary imposition on the notation originally reserved for cardinal multiplication. Noncounting meanings of juxtaposition arise quite naturally and _immanently_ out of ordinary-number algebra, assumed at first to involve only Real numbers. For example, in $x^2-4 = 0$, x stands for a counting operator[†] or 'countor', namely 2, which 'counts itself' ("two _of_ two'), yielding 4: xx = 4. Here (self-)juxtaposition does indicate "multiplication" or the "times" interaction. However, when we come to equations like $x^2+1 = 0$, something different is going on. This equation implies that xx or x(x) = -1. In this case, juxtaposition of x to x denotes a different operation entirely. That is because the solution to this equation, i, is a different _kind_ of number altogether from "Real" numbers. The square root of minus one, i, does not count, it _rotates_. i operating on i is a movement from the i position to the -1 position in the complex plane, via a 90° counterclockwise circular rotation.[*]

In summary, as the foregoing example attests, the operational meaning of juxtaposition is controlled by the operational meanings of the symbols juxtaposed. We can dispense with any intervening signs of "pure operation".[30]

Processes, Rectilinear and Circular

At this point, the connection with the problem of nonlinearity is ready to emerge. So far, in the form 'ab → c', we have considered the image of what I shall call _rectilinear_ _process_ only. In such a process, _a_ as a subject acts upon _b_ as object, yielding _c_ as result, wherein

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[*] [See PO & LC, pp. 27 ff.; pp. 33, 43]

[†] [We are ignoring the other root: x=-2. On the operatorial nature of negative Real numbers, see PO&LC pp. 32-33.]

$$a \neq b \neq c \neq a$$

That is, all the component processes are distinct. We start at a and b, and wind up somewhere else, c, as if moving in a straight or rectilinear path. In economic terms, for example, this corresponds to a model of production-distribution which begins at "one end" of the economic process, with "inputs" or "ingredients", and stops describing at the "other end"; the "outputs" or "final products". The inputs are treated as a category quite distinct from the outputs. A sentence-formula for this model might read as follows:

> 'Factors of production interact, yielding consumption goods.'
>
> 'Labor and means of production act upon raw materials, producing consumption goods.'

However, a second fundamental process-paradigm is implicit in the form '$ab \rightarrow c$'. It corresponds to the case

$$a = b = c,$$

the case of "self-interaction" of a process:

$$aa \rightarrow a$$
$$a(a) \rightarrow a$$
$$a^2 \rightarrow a$$

Form (2) characterizes what I shall call circular process.

Pierro Sraffa in 1960 published "Production of Commodities By Means of Commodities", whose very title suggests the kind of process we are now considering. This little gem of a work--less than 100 pages--seeded a tremendous outpouring of new economic thinking and writing, a veritable Renaissance of "pre-neoclassical" economic theory, which continues to this day. Sraffa characterized his approach in this work as follows: "Its approach is reminiscent of certain points of view taken by the classical economists from Smith to Ricardo but neglected in the period of domination of the 'marginal method'. One of the returns to earlier ways of thought here adopted is...to regard production as a circular process in which the same kind of commodities appear both among the means of production and among the products--rather than as a process beginning with 'factors of production' and ending with consumption goods."[31]

A sentence-model of the above conception might read:

'The economy produces the economy.'

Wherein the subject of the sentence, 'the economy' (which is also its object), is meant to comprehend the entire process of economic life, including especially the human agents in their cooperative interconnection, and not just commodities, machinery and raw materials--not just "things". The sentence-model expresses the concept we shall call "economic reproduction".

A more comprehensive rendition of "economic reproduction" might run:

'The economy, by means of the economy, produces the economy.'

or

'The economy, interacting with(in) itself, reproduces itself.'

or

'The economy acts upon the economy yielding the economy again.'

or

'The economy, using itself (plus portions of
extra-economic Nature) as means of
production, reproduces itself.'

We will consider the problem of the 'extra-economic Nature' connection in the "Boundaries of the Economy" subsection, below. Leaving that aside for now, the above formulas present the input, the output, and the 'processor' of the economic process as one and the same entity, "the economy". Thus, if we take $\underline{a}$ to stand for the process called "the economy", then the Sraffa approach invokes the form '$aa \to a$': "$\underline{a}$ of $\underline{a}$ yields $\underline{a}$".

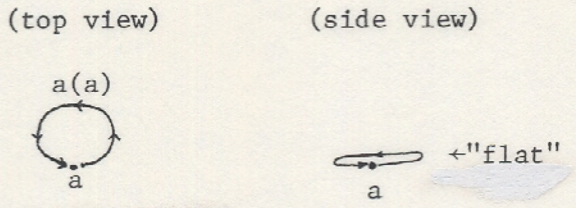
The "circular" model (2) is thus the very paradigm for this "reproduction" conception of the economic process. This is a model of "simple" reproduction. That is, the "self-interaction" of the economy leads back to the original state of the economy. The economy reproduces itself without change: without growth or decline; without any qualitative metamorphosis. This idealization is descriptively inadequate for "live" economies. In particular, it cannot encompass "crises", points of disturbance in the quantitative growth pattern, and of qualitative repatterning of growth [see Subsection 2, "A Working Definition of Economic Crisis"]. We really want to model, not "simple" circularity of process, or simple return to a starting point (state), but what we might designate "helical circularity", involving a developmental evolutionary progression

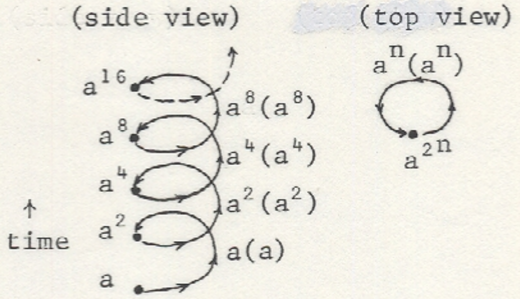
$$a \to (a)(a) \to (aa)(aa) \to (aaaa)(aaaa) \to \dots$$

such that

$$a \neq a(a) \neq a^2(a^2) \neq a^4(a^4) \neq \dots$$

This helical model would preserve a circular or "cyclical" aspect of periodic recurrence of qualities. But recurrence would arise each time at a different level in the progression:





SIMPLE CIRCULARITY

$[a(a) \rightarrow a]$

$a = a(a) = a^2(a^2) = a^4(a^4) = \ldots$

HELICAL CIRCULARITY

$[a \rightarrow a^2 \rightarrow a^4 \rightarrow a^8 \rightarrow \ldots]$

$a \neq a(a) \neq a^2(a^2) \neq a^4(a^4) \neq \ldots$

An expression of the form 'e(e)', denoting the self-interaction of a process, we will term a "reflexion" or "reflexive function"*. On first take, the term "self-interaction" may seem a misnomer. The "inter" implies something goes on *between* the process *e and itself*. But the idea of an interaction "between *e* and *e*" is actually descriptive of the concept of 'reflexion'. For it suggests the action of *e as subject* upon *e as object*. That is, it suggests an interplay between the *subjective* (or action-initiating) and the *objective* (or action-receiving) *aspects* of one and the same entity.

Nonlinearity in Light of Operator Ideogrammar

What has all this to do with nonlinearity? Nonlinearity essentially involves higher degree terms, the higher *powers of an operator*, that is *self-operation*. Let's look at the general form of a nonlinear term in differential equations. Suppose we don't know the function (operator) that tells the *state* for every time t of the system we are modeling. That function, explicitly rendered, is the "mathematical model" which we seek. But suppose we do know something about how the system changes with time. Suppose we formulate that knowledge as a differential equation. The state-function is then the "unknown" of that equation. Call this function x. Then x(t) or (xt) stands for the values of that function; the result of operation upon t. That is, x(t) stands for the state of the system at time t. If D stands for the operation "differentiation with respect to time" then our differential equation takes the form:

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*[In this case, of the "static" type. Reflexive functions of the "dynamical" type will come up further on.]

$$D^m x(t) = \ldots + Ax(t)^n + \ldots \qquad (3)$$

wherein $Ax(t)^n$ represents a typical term of the whole expression for the mth time-derivative of x(t). Suppose n is an integer greater than +1. Then (3) is a nonlinear differential equation. It states that the value measuring the rate of change of system-state per unit time^m, denoted $D^m(xt)$, is the same number as that resulting from the operation of the constant A upon the function-value xt operating upon itself n-fold (plus other terms). Suppose we nickname xt by X. Suppose A = 1. Then the nonlinearity of (3) is revealed to be a matter of the self-functioning or self-operation of the operator X = xt:

$$X^n = \underbrace{X \cdot X \cdot \ldots \cdot X}_{\text{n occurrences}} = \underbrace{XX \ldots X}_{\text{n presences}} = \underbrace{X(X(X \ldots (X) \ldots))}_{\text{n occurrences}}$$

In the context of a differential equation, nonlinearity means that the rate of evolution of the system is proportional to the state-value itself, X, self-multiplied. This suggests that the evolution of a nonlinear system is caused by the self-operation of each of its states; that is, by the self-interaction of the system.* By self-operation of a system we mean the "self-refluxion" (return flow to source) and "self-reflexion" of its activity-states, a concept which requires considerable clarification.

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*[See PO & LC, pp. 18-20.]

For this clarification, the "grammatical" and ontological principle of noun-verb synthesis becomes crucial. We conceive a system, on a noun-like model, as an essentially inert "being" *to which* actions and "becomings" happen--which "goes through" changes [thus conceived as essentially *external* to its substance and identity]--and which, in addition, perhaps, episodically enters into the realm of action on its own initiative, by committing acts or causing actions. The system thereby, linguistically, *momentarily* attaches to its name certain process-names, *verbs* which, however, belong to an essentially separate realm, and which form no part of the system's inner essence, substance or identity. For such a model, the idea of self-action and consequent self-change borders upon the unintelligible, or at least the paradoxical. For such a system, "self-interaction" would be an odd occurrence. The idea of such a thing *doing something to itself* seems somewhat far fetched, since it is not even essential to its existence that it do anything to anything *else*. The idea that, simply by "being" it is *continuously* "doing to itself", acting upon itself, is out of the question.

But if we conceive a system *from the first* as a process, as a process-*entity* or "eventity", whose being *is* activity, it is otherwise. This verb-like model presents the system as an inherently active, *substantially* dynamic "being". In fact, it presents a "becoming", and not a "being" at all, as "being" is habitually imagined under the shadow of our modern language, with its dualistic noun-verb ontology and its "verb" *to be*. With such a "verbal" model in mind, the idea that an entity affects itself and changes itself *simply by being* becomes almost obvious.

Grasping the function-value or state-value xt = X as an *operator* allows us to interpret differential-equation nonlinearity in this light. Such nonlinearity means that every state (re)acts back upon itself, thus inducing a movement out of that state, a negation of that state--a change of state. Each state implies its own negation. Each state causes its own negation, negates itself. The occurrence of a state contradicts its continuance.

The continuum of the self-negation of each state is the state space trajectory, or course of evolution, of the system. This trajectory expresses the evolutionary self-propulsion of the system, its self-development via the reflexion of action upon the source of action.* Concrete forms of this self-reflexion of action, in the domain of economic systems, are elaborated below.

Nonlinearity In Logic

An enhanced grasp of this definition of nonlinearity may be had by recourse to the realm of logic whose subject matter is, or should be, even more directly conceptual than that of mathematics proper. A central unsolved problem of that realm is that of the "reflexive paradoxes". That these paradoxes are the logical analogues of the unsolved nonlinear equations and the uncharted nonlinear functions and operations of mathematics is not commonly recognized. Its recognition aids enormously in clarifying the problem of nonlinearity and identifying its full conceptual generality. That generality encompasses far more than narrowly technical matters confined exclusively to mathematics. The paradoxes reveal those junctures in the realm of the logic of nouns where the inescapable intrusions of the logic of verbs are felt. Every one of these paradoxes is associated with the internal breakdown or transcendence of a duality in the syntactical and ontological classification of symbolized logical objects. This internal breakdown is the precise analog, in formal logic, of the supersession of the noun/verb dichotomy in natural language, or of the quantity/operation and function/argument dichotomies in mathematical language, to which we have traced the difficulties with nonlinearity. That is, the "reflexive paradoxes" are examples of "reflexive functions" of the type which arise *within* formal logic.

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*[See PO & LC, pp. 17-20.]

Take the oldest of these, the 'pseudomenon' of Epimenides, the paradox of the self-denying sentence, the self-negating proposition:

This sentence is false. (4)

Here we have a breakdown in the distinction between propositions and the "world of objects" to which they refer. The formal object or referant of the above proposition is the proposition itself. The tacit notion that propositions *merely describe* the objective world, passively, without affecting it or interacting with it, and as if *from outside that world*, does not hold here. The proposition becomes an object, a part of the objective world. The proposition or sentence is in reality a *thought*, a *mental* object. The *written* sentence, the *symbol* representing that actual, conceptual sentence or proposition, is a code, a *prescription* or *program* for thought. When the prescription of the sentence-symbol (4) is mentally carried out, it appears that the sentence continually acts upon itself, perpetually changing its truth-value from F to T back to F again. If you accept the assertion, and think that the sentence is false, that makes it true, which in turn makes it false, and so on. Thus, this proposition apparently generates a sustained *oscillation* in truth-value.

In the attempt to express this sentence in symbolic logic, the following telling notational phenomenon arises. Symbolize by F^1 the predicate (unary relation) "falseness". Let T^1 denote the predicate "trueness". Let x be an individual constant standing for the object of which trueness or falseness is to be asserted. Thus x here stands for the sentence (4). Then F^1x means "$x \epsilon F^1$" ("x is an element of the set of all false things".) It stands for the sentence "x is false". But x is the sentence "This sentence is false": $x = F^1x$. Hence we can substitute F^1x for x wherever it occurs, obtaining:

$$F^1x = F^1(F^1x) = F^1(F^1(F^1x)) = F^1(F^1(F^1(F^1x))) = "(F^1)^{\infty}",$$

and we get the contradiction:

$$F^1x \leftrightarrow F^1(F^1x)$$

[translation: 'This sentence is false' is false implies 'This sentence is false' is false' is false.] or

$$F^1x \leftrightarrow T^1x$$

$$[\rightarrow T = F;\ \text{True} = \text{False}]^*$$

Or take the Grelling paradox. This involves the predicate "heterological", which we will symbolize by H^1. 'Heterological' is a predicate about predicates. It describes a predicate or adjective which is not applicable to itself, not true of itself. The contrary of heterological is autological, which describes predicates which are self-descriptive. 'English', 'adjectival', 'pollysyllabic', 'conceptual', 'phonetic', and 'verbal' are examples of autological predicates. 'French', 'adverbial', 'monosyllabic', 'nonsymbolic', 'pictographic' and 'tactile' are examples of heterological predicates. Since these two predicates apparently divide the class of all predicates between them, the question naturally arises "into which of these two divisions does each of them fall?" The paradox is precipitated by the question: "Is heterological heterological?" Does 'heterological' apply to itself or not? If it does, it doesn't, because applying to itself makes it autological. If it doesn't, it does, because non-self-applicability is the meaning of 'heterological'. Again, sustained

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[*] [Here "↔" stands for biconditional implication, or the relation "iff" (if and only if). The single-headed arrow "→" means "implies" or "if...then...".]

oscillation. In the ideographic symbolism of the first-order predicate calculus, with x an individual constant standing for the predicate heterological itself -- $H^1 \equiv x$ -- We get

$$H^1x \leftrightarrow \sim H^1x \qquad \text{or simply } H^1H^1 \leftrightarrow \sim H^1H^1$$

$$H^1x = H^1H^1 = (H^1)^{"2"}$$

Again, paradox arises where a grammatical and ontological division is transcended. Here, the distinction between relations and predicates, denoted by capital letters (H^1, F^1, etc.), and "logical individuals" or "objects", denoted by lower case letters, such as x, is abrogated. $H^1 = x$; H^1 is both "predicate" (or "unary relation") and "individual".

If we denote by y the proposition "heterological is heterological", symbolized by H^1x or H^1H^1, so that $y = H^1H^1$, we get the formula of formal contradiction familiar from the previous example

$$T^1y \leftrightarrow F^1y \rightarrow (T = F)$$

$$F^1y \leftrightarrow F^1(F^1y)$$

$$F^1(H^1H^1) \leftrightarrow F^1(F^1(H^1H^1))$$

Note that, if we interpret H^1 extensionally, so that H^1 denotes the set of all heterological predicates, then

$$H^1x \rightarrow x \in H^1 \rightarrow H^1 \in H^1$$

invoking a set which is among its own elements; self-membership of sets. If we symbolize "monosyllabic" M^1, "pictographic" P^1, "adverbial" A^1 and so on, we have

$$H^1 = \{\ldots, M^1, P^1, A^1, H^1, \ldots\} = \{\ldots, M^1, P^1, A^1, \{\ldots M^1, P^1, A^1, H^1, \ldots\}, \ldots\}$$

$$= \{\ldots, M^1, P^1, A^1, \{\ldots, M^1, P^1, A^1, \{\ldots, M^1, P^1, A^1, H^1, \ldots\}, \ldots\}, \ldots\} = \ldots$$

Set self-containment entails infinite depth of containment.

The exotic conceptual phenomena of set self-containment form also the axis about which turns the most recalcitrant and devastating of the paradoxes, the Russell paradox. That is, this paradox hinges on a transcendence of the distinction between sets and elements or members of sets. Russell has us consider the set of all sets which are not members of themselves. He then asks whether this set is a member of itself. The conclusion?: if it is, it isn't, and if it isn't, it is. Again, paradox presents the form of logical oscillation, an oscillation between the two truth-values that sustains itself indefinitely, that can never come to rest at the one truth-value or the other. In terms of our actual, time-taking mental process as we think through the argument, the Russell set takes on a pulsating image. It alternately expands and contracts in membership, first ingesting itself, only to disgorge itself a little later, then ingest itself again.

In predicate symbols, using "$\notin^2$" for the relation $p \notin p$ (where p denotes a set) and ε^2 for $\sim\notin^2$, the relation $p \varepsilon p$, and if we again define $\notin^2$ and ε^2 extensionally, so that $\notin^2$ equals the set of all sets which are not members of themselves, we get, taking x to be the individual constant denoting the set $\notin^2$ $(x = \notin^2)$:

$$\notin^2 xx \leftrightarrow \sim\notin^2 xx \quad ; \quad \notin^2 xx \leftrightarrow \varepsilon^2 xx,$$

or: $$\notin^2\notin^2\notin^2 \leftrightarrow \sim\notin^2\notin^2\notin^2 \quad ; \quad \notin^2\notin^2\notin^2 \leftrightarrow \varepsilon^2\notin^2\notin^2,$$

that is: $$(\notin^2)^{\text{"3"}} \leftrightarrow \sim(\notin^2)^{\text{"3"}} \quad ; \quad (\notin^2)^{\text{"3"}} \leftrightarrow \varepsilon^2(\notin^2)^{\text{"2"}}.$$

In The Principles of Mathematics, Russell explicitly notes the connection of this paradox to mathematical nonlinearity. In terms of his "propositional function" notation for symbolic logic, this paradox takes the form $\phi\{f(\phi)\}$ -- "the argument itself is a function of the propositional function".[32] He likens such logical expressions to quadratically nonlinear terms in ordinary algebra.

Thus the phenomena of propositions about propositions, predicates about predicates, sets of sets, etc. seem to give rise to a nonlinear logic. We might call such mental objects logical 'meta-objects', "objects of second degree". That is, a proposition about a proposition is a proposition "of second degree,"; a predicate describing predicates is a "quadratic" predicate; a set of sets is "a set squared", and so on. Informally, but suggestively, we might write:

$$\text{sets}(\text{sets}) = \text{sets}^2.$$

Such objects give rise to a "logic of second degree".[33] At certain junctures, this nonlinear logic gives rise to the paradoxes. Nonlinear logic is therefore the logic of paradox. Or, perhaps more accurately, certain of the conceptual phenomena of nonlinear logic are paradoxical for the old habits of thought, adapted exclusively to "linear logic". By "linear logic" is meant the logic of first degree, treating of propositions about objects not including propositions, predicates about objects not including predicates, and sets of objects not including sets, etc.

In Principia Mathematica Russell identifies circularity and self-reflexivity as the causes of the paradoxes of set theory and formal logic:

> In all of the above contradictions (which are merely selections from an indefinite number) there is a common characteristic which we may describe as self-reference or reflexiveness.[34]

and

> An analysis of the paradoxes to be avoided shows that they all result from a certain kind of vicious circle.[35]

The central point of this subsection is that logical (and material) *circularity* need not be "vicious" or meaningless; that such circularity can be quite useful, appropriate and even necessary for modeling economic reality,* providing the essential paradigm for comprehending the phenomena associated with economic *nonlinearity*.

In her *An Introduction to Symbolic Logic*, Susan K. Langer, noted student of Russell, inadvertently makes succinctly explicit the real source of the "paradoxicalness" of nonlinear logic[36]:

> Because of the shortcomings of natural language, logicians adopt the simpler and more consistent medium of ideographic symbols...Ideographic signs may be freely adopted within the limits of the following principles of symbolic expression: (I) *Radical distinction between term signs* [like our x] *and relation signs* [like our $\notin^2$ and H^1]. [emphasis added]

This "radical distinction" is the same one which Whorf identified as the legacy, in ideographic symbolic logic, of the noun-verb dichotomy of later Sanskrit and Indo-European grammar.

"Relations" and "predicates" are the *staticized*, detemporalized, logical versions of verbs. In formal logic, which we could characterize as "the logic of nouns", a statement like "The sun *shines*" becomes "The sun *is* an element of the set of all shining things": $s \varepsilon S^1$; $S^1 s$.[37] In our examples, paradox arises where a "logical verb", or an entire proposition--a relation or predicate, or an entire sentence--demands to be treated also as a "logical noun", i.e. as an object or "logical individual" which can enter into sets as an element and about which relations, predicates, and propositions can be asserted. The "contradiction" posed by such situations is, *at root*, a contradiction of the premise presumed at the outset, namely that "terms" and "relations", "logical nouns" and "logical verbs" are radically, ontologically dirempt. This contradiction of initial assumption arises quite naturally, immanently, in the development of the logical system from that very premise. This initial assumption of a radical diremption between "terms" and "relations" is not sufficient to exclude certain extremities, the paradoxes among them, where the two become one. The premise of "noun-verb dichotomy" is thus, in a sense, inconsistent, self-contradictory. The development from it, at length reaching logical objects of higher degree, including the "paradoxical" objects we have reviewed, strikes back at it, refuting it.

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*[See quoted passage and discussion below, pp. 143-147.]

The emergence of logical objects of higher degree within formal logic leads, in certain cases, to paradox. But this is not so much because such "self-reflexive" occasions are inherently unintelligible in themselves, as because they contradict the ontological presuppositions upon which the entire grammatical and syntactical set up of the artificial *language* of logical ideography had been constructed. The reflexive occasions contradict, not logic *per se*, but the grammatical structure and ontological assumptions of *this particular* ideographic representation of ideas.

Relax these assumptions, abandon the dualistic ontology they enforce, and a "logic of verbs"--or better, a logic of noun-verb unities, a logic of "*eventities*"--results. Logical nonlinearity loses its apparently destructive and anomalous character, and can be *utilized constructively*.[38] The paradoxes dissolve *as such*, and turn into *models* and *paradigms* for a logic of reflexive processes.

Implementation of a logic of verbs, activation of the latent "verb" aspects of logical objects, involves the emergence of *dynamical* and, as we shall see, *temporal* features *within* logic. *Time* and *motion* acquire a logical status. Note the inherently *dynamic* character of the paradoxes we reviewed--the pulsating Russell set, for example. The paradoxes in general represent "ideas which won't stand still" in the mind--logical models of self-induced motion.

A profound connection between logic and mathematics was intuitively felt long before Russell and Whitehead attempted, in Principia Mathematica, to deduce all of mathematics from formal logic alone. We can show that classical linear logic is intimately related to linear mathematics in general, to linear systems, and to linear differential equations. We can also show that "nonlinear logic", the logic of eventities, is profoundly connected to nonlinear mathematics in general, nonlinear systems, and nonlinear differential equations. In particular, this logic is related to the "nonlinear algebra" of the Musean hypernumbers, recently discovered operators which give, for the first time, concrete hope of achieving exact solutions to hitherto intractable nonlinear integrodifferential equations; solutions in explicit, closed form.

The objects of classical linear logic, logical objects of the "first degree", are "equilibrium systems" in logical space. If they start out with the value "true" they stay true, if "false" they stay false. The nonlinear and paradoxical objects, the logical objects "of higher degree", represent "nonequilibrium systems", oscillatory logical processes that never come to rest at one truth-value or the other. The paradoxes are cases of self-excited oscillation in logical state. They are logical self-oscillators.

The terms "self-excited oscillation" and "self-oscillator" are well known in another realm. As highlighted in recent work by Dr. Ilya Prigogine et al.[39] nonlinear differential equations typically have as solutions state-functions representing nonequilibrium systems. Depicted geometrically, as trajectories in the state-space of the equations, these solutions typically take the form of "limit-cycles" representing indefinitely sustained self-excited oscillations in system-state. Dynamical systems with limit-cycle behavior are known as "self-oscillators". Linear differential equations, on the contrary, typically describe equilibrium systems, represented by fixed points, or stationary positions in the state space. These fixed points denote a final fixed state toward which all possible system trajectories are attracted.[40]

These observations provide clues to the logic-mathematics connection. They suggest that linear logic, with its fixed points in truth-value space, or "truth-state-space", is the logic of equilibrium. Dealing with movements that have already come to permanent rest, with unchanging states, linear logic can dispense with concepts of time and motion. Nonlinear logic, on the contrary, means a nonequilibrium logic, a logic of limit-cycles, a logic of oscillation and of "standing-wave" vibratory entities in general.

Long before Russell and Whitehead attempted to reduce mathematics to logic, George Boole attempted the inverse programme of "reducing" logic to mathematics. In The Mathematical Analysis of Logic and The Laws of Thought, Boole attempted to capture the thought-process in ideographic symbols. "The Boolean algebra" in use today is a far cry both from the "calculus of mental operations" which was Boole's stated goal, and from the original system with which he sought to approach it. Boole intended to implement only classical Aristotelian logic, "linear logic". But the power of the means he discovered seems to have augured, at certain critical junctures in his system, something more. The operators he required to inaugurate this ideography were simply 0 and 1. Boole modeled logic by an arithmetic confined to the unit interval. It is interesting to translate the general form of the paradoxes into original Boolean. The manner in which their oscillatory essence emerges in this system in connection with others of its fundamental formulae, gives rich insight into the logic of nonlinear differential equations, and, I believe, a clue to their explicit solution.

In the "secondary" or propositional interpretation of Boole's calculus, the compound proposition "If Y^0 is true, then X^0 is true", wherein X^0 and Y^0 are themselves propositions (sentences), is symbolized as follows:[41]

$$y = \frac{0}{0}(x) \qquad \text{[in modern symbolic logic: } Y^0 \rightarrow X^0\text{]} \qquad (5)$$

where the quantifying operator $\frac{0}{0}$ represents an indeterminate logical quantity--all, some, or none. Time enters explicitly into the Boolean version of propositional logic. The value of y measures the time during which the proposition Y^0 is true, the value of x expresses how long the proposition X^0 holds true. Ordinarily, the two admissable cases are $y = 0$, meaning Y^0 is true never, none of the time, and $y = 1$, meaning Y^0 is true forever, all of the time. The value 1 stands for "eternity", eternal truth, 0 for no time at all, eternal falsity. The value $\frac{0}{0}$ can stand for "some" of the time--neither all nor none of the time--but Boole ordinarily disallows it as an explicit solution-value.

Let Y be a sentential letter standing for the precipitating proposition of any of the paradoxes. Thus, Y^0 could denote the sentence "'This sentence is false' is true" or "'heterological' is heterological" or "The Russell set is a member of itself". Let X^0 denote $\sim Y^0$. That is, X^0 might stand for "'This sentence is false' is false", or "'heterological' is not heterological" or "The Russell set is not self-containing". In Boolean, not-Y^0 is written (1-y). This signifies that a proposition whose truth lasts for a duration y, has a negation true for a duration 1-y, or for all time less the amount of time y. In this way we will formulate the general form of the paradoxes, the form of logical self-contradiction:

$$Y^0 \leftrightarrow \sim Y^0$$

or

$$T^1x \leftrightarrow F^1x$$

where x stands for the paradoxical proposition (i.e. where $x \equiv Y^0$). In Boolean:

$$y = \frac{0}{0}(1-y)$$

To solve equations of the form (5), Boole first eliminates $\frac{0}{0}$. He transposes (5) to a form equal zero. For our special case:

$$y - \frac{0}{0}(1-y) = 0 \tag{6}$$

He then regards (6) as a function of $\frac{0}{0}$. Call ours $f(\frac{0}{0}, y)$. He proves that, whatever f or y may be, $f(0,y) \cdot f(1,y)$ must equal zero.[42] 1 and 0, recall, were the two admissable, explicit values for $\frac{0}{0}$. For us

$$\frac{0}{0} = 1 \text{ implies } f(1,y) = y-1(1-y) = 0 \quad \text{or} \quad y-(1-y) = 0 \quad \text{or} \quad 2y-1 = 0$$

$$\frac{0}{0} = 0 \text{ implies } f(0,y) = y-0(1-y) = 0 \quad \text{or} \quad y = 0$$

The equation '$f(0,y) \cdot f(1,y) = 0$' becomes in our example

$$(y) \cdot (y-(1-y)) = 0$$

$$y^2 - y + y^2 = 0$$

$$2y^2 - y = 0$$

whose roots, by the quadratic formula, are

$$\frac{+1 + \sqrt{1-0}}{4} \quad \text{and} \quad \frac{+1 - \sqrt{1-0}}{4} \qquad \text{i.e. } \frac{1}{2} \text{ and } 0.$$

Actually, we could have simply solved $y - (1-y) = 0$ or $2y - 1 = 0$ to get $y = \frac{1}{2}$. The solution $y = 0$ corresponds to an assertion that the paradoxical proposition Y^0 is simply false. The solution $y = \frac{1}{2}$ is more peculiar. Interpreted literally, it means that Y^0 is true $\frac{1}{2}$ the time. Then $\sim Y^0$ would be true $1 - \frac{1}{2} = \frac{1}{2}$ the time also. Y^0 is true half the time and false half the time. So is $\sim Y^0$. What better static characterization of an endless alternation in truth-value could be imagined![43]

But Boole would never have allowed it. True, the value $\frac{1}{2}$ falls within the unit interval of the Real number-line, whose properties he used as a metaphor for formal logic. However, the operator $\frac{1}{2}$ disobeys that crucial formula which governs all operators usable to model formal logic, and which Boole dubbed the "Fundamental Law of Thought":[44]

$$x^2 = x \tag{7}$$

(whereas $(\frac{1}{2})^2 \neq \frac{1}{2}$). Given in this form this equation simply asserts that logical nonlinearity reduces to linearity. In the "primary" or existential interpretation of Boole's system, symbols like x represent "concepts" extensionally. They denote sets of objects having a quality in common. For example, the concept of 'greenness' might be denoted by way of g, where g would connote the set of all green things. The value of g would represent the size of the set of green things, its "cardinality", in the following sense. If all green things in the universe (of discourse) were green, then g = 1. If no things were green, g = 0. The latter would mean the set of green things had zero elements, i.e. the nonexistence of the concept "green-ness" for the universe of discourse in question. If only some things were green, this could not be expressed directly [except in rare cases, by the indeterminate value $g = \frac{0}{0}$]. In the "primary interpretation" then $x^2 = x$ means that the self-interaction of a concept, x, is without effect. Boolean "multiplication" models set intersection. The self-intersection of a set is just the original set unchanged. For these reasons, the form (7) can be grasped as a strengthened version of the "law of identity" of Aristotelian logic: A = A, or, in the propositional realm, $\mathbf{P}^0 \leftrightarrow P^0$. Transposed and factored, (7) reveals that an expression for the classical "law of noncontradiction" was hidden within the form of static self-identity:

$$x - x^2 = 0$$

$$x(1-x) = 0 \qquad (8)$$

Any concept x and its negation (1-x) have nothing in common; the empty set as intersection. In modern symbols this translates to "$x \cap \overline{x} = \emptyset$" or, in propositional interpretation, "$\sim(P^0 \ \& \sim P^0)$" or $(P^0 \ \& \sim P^0) = F$ (wherein F stands for the truth-value "false"). Boole interprets (8) as a description of the "fundamental" modus operandi of our mental process: analysis and classification into pairs of opposites, or "dichotomy". For this reason he calls the "Fundamental Law of Thought" also the "law of duality".[45]

If we "sum" the two factors (x) and (1-x), instead of "interacting" ("multiplying") them, we get the classical "law of the excluded middle". For the Boolean '+' operation corresponds to set union, or the logical "exclusive or" function:

$$(x) + (1-x) = 1$$

In modern symbols, "$x \cup \overline{x} = u$" (wherein u = the universal set or set of all objects) and, propositionally, "$(P^0 \vee \sim P^0)$" = T (wherein T stands for the logical-value "True").

Thus Boole's "Fundamental Law" encompasses the three laws of classical logic in a single formula. In his original presentation of the system, the 1847 Mathematical Analysis of Logic, Boole gave this law in a more general form:[46] $x^n = x$. This form expressed the reduction of logical nonlinearity of all degrees to the first or linear degree. But by the publication of the 1854 Laws of Thought, difficulties had emerged which led Boole to restrict the Fundamental Law to the form (7).

He noted that (7) was an equation "of second degree", a nonlinear equation--the only such allowed in his system. To the degree 2 corresponded the dichotomizing, dualistic mode of cognition he posited as essential to deductive thought. He complained that allowing n in $x^n = x$ to take values other than 2 brought in new factors, paradoxical or "not interpretable" with respect to formal logic. For example, n = 3 gave

$$x^3 = x$$

$$x - x^3 = 0$$

$$x(1-x)(1+x) = 0 \qquad \text{or} \qquad x(1-x)(-1-x) = 0 \tag{9}$$

involving a third root, $x = -1$, which disobeys $x^2 = x$, and a third factor $1 + x$ or $(-1-x)$. He could manage no logical interpretation for -1. Nor was "the addition of a class x to the universe", $(1+x)$--the expansion of the universe of discourse--conceivable. And the threefold factorization (9) of $x-x^3 = 0$, suggested a procedure of "trichotomy" instead of dichotomy, a "threefold mental division" and classification operation upon the object of thought which is: "of a nature altogether foreign to the province of general reasoning" and which "it is impossible for us, with our existing faculties, adequately to conceive."[47]

Of course, 0 and 1 remain roots of $x^n = x$ for all positive integral values of n. But other roots, indeed, "creep in". These numbers x make $x^n = x$ a true equation for a given value of $n > 2$. But for $n = 2$, these x's are such that $x^n = x$ does not hold true for them. These operators do not observe the "Fundamental Law" of classical logic. In particular we have the following newly discovered roots which do not obey $x^2 = x$:

$$x^3 = x; \quad \text{wherein } x = \pm\varepsilon \text{ and } (\pm\varepsilon)^2 = x^2 = +1, \text{ but } \pm\varepsilon \neq 1$$

$$x^7 = x; \quad \text{wherein } x = \pm w \text{ and } (\pm w)^2 = x^2 = -1\pm w$$

$$x^9 = x; \quad \text{wherein } x = \pm p \text{ and } (\pm p)^2 = x^2 = 0, \text{ but } \pm p \neq 0$$

to which we can add

$$x^5 = x, \quad x = \pm i, \quad (\pm i)^2 = x^2 = -1.$$

These "roots"--denoted ε, w, p, and i--are Musean hypernumbers[48], of which +i and -i, the square roots of negative Real unity, are the only presently well-known examples.

These hypernumbers solve $x^n = x$ for specific cases of n, for a reason very different from that which allows 0 and 1 to solve $x^2 = x$ and indeed any equation of the form $x^n = x$, for n any positive integer. Both zero and 1 represent fixed points in number space. The operation of 0 upon itself n times, 0^n, or of 1 upon itself n times, 1^n, produces no movement, no trajectory. In other words, the whole trajectory is a single point.*

Hypernumber Logic

With i and the other hypernumbers cited, however, it is otherwise. It is not only that their repeated self-operations generate true trajectories. They solve equations of the form $x^n = x$ because, in each case, their n-fold self-operation generates a self-connecting trajectory, a closed orbit. Because this orbit is generated by the powers of an operator, it is called the "power-orbit" of the operator.[49] The equation $x^n = x$ is simply an expression of that orbit, an algebraic, formulaic description of that closure. It states that, after operating upon itself n-1 times, the operation, x, has returned to its original starting position, x^1; that x^1 and x^n coincide. Each of the hypernumbers has a different, characteristic value (or values) of n for which this is true. For $\sqrt[2]{-1}$, this n is 5 (and any integral power thereof). Five-fold self-reflexions of i generate a circular movement in the complex plane[*]:

---

*[see PO & LC, p. 46]

$$i^5 = iiiii = i(i(i(i(i)))) = i^1 = i$$

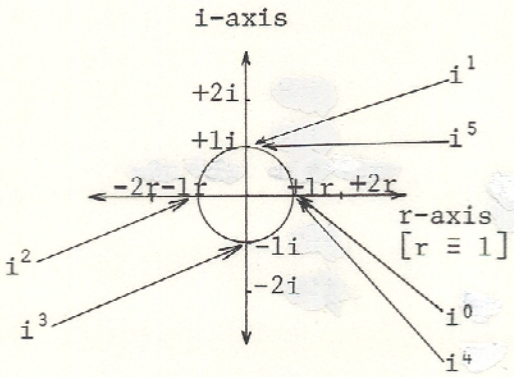


Figure 1: the i power-orbit, illustrating $i^5 = i$

If the radical grammatical dualism of "noun" versus "verb" seeds our conceptual difficulty with nonlinearity, our realization of mathematics to date reflects this process. The "Real" numbers, the operators we have chiefly used and developed until recently, obscure the importance of self-interaction. The unity of the Real numbers, the operator r or +1, is impervious to self-reflexion. It is the "identity" operator. Self-operation, repeated no matter how many times n, yields no change, no movement at all. r sends r right back where it started. Or rather, r never leaves. It simply stays in place. Boole's "Fundamental Law", $x^2 = x$, is just a description of this stasis.

$$1(1(1(\ldots(1)\ldots) \to 1$$

$$1^n \to 1$$

---

[*] [See PO & LC, pp. 38-39, 43.]

Multiple forms of r represent rectilinear movement, along any axis--along the "adopted" axis of the operand.* Thus $+100(-2r) \rightarrow -200r = -200$, simply a linear movement further down the negative branch of the Real axis. Similarly, $+100(+2r) \rightarrow +200r = +200$, further down the positive branch. Likewise $+100(-2i) \rightarrow -200i$ and $+100(+2i) \rightarrow +200i$, movements confined to the i-axis, the i number-line.

By contrast, all but one of the hypernumber operators present nonrectilinear trajectories, curves, as the characteristic geometrical expression of their action in number-space. The "signature" of the i operation is a 90° circular rotation, as we have seen. The "signature" of ε is also circular, but with a quite different "attitude" with respect to the ε and r axes and the origin. The w process is an elliptical rotation. The p process describes a lobate form.

The Logic of Differential Equations

The "fixed point" character of 1^n expresses the logic of that operation, the logic of r. Boole's $x^2 = x$ captures the fundamental law of that logic, i.e. of Aristotelian or linear logic. In The Mathematical Analysis of Logic, Boole explicitly noted the connection between the logic that flows from $x^2 = x$ and the logic of linear differential equations. For reasons we won't go into, he refered to the symbols of his logical algebra as "elective symbols":

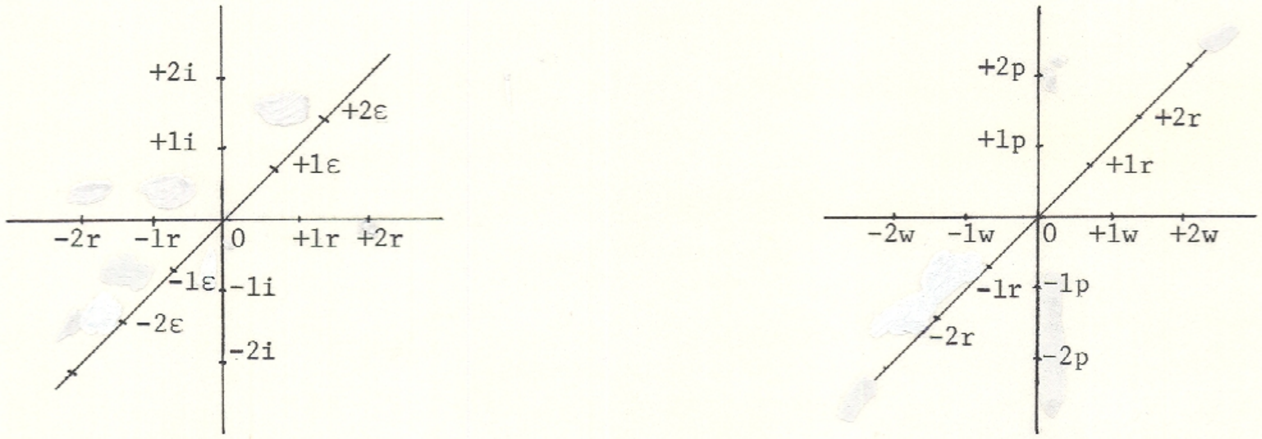
> It is to be observed of such equations, that from the very nature of elective symbols, they are necessarily linear, and that their solutions have a very close analogy with those of linear differential equations...[50]

Thus nonparadoxical or linear logical expressions correspond to linear differential equations. We have seen the correspondences between paradoxical or "higher degree" logical expressions, like $y^2-(y-y^2) = 0$, and nonlinear differential equations in terms of "sustained oscillation" solutions. Solutions like $y = \frac{1}{2}$ of such logical equations do not observe the $x^2 = x$ law. The hypernumbers are among these solutions to higher degree logical equations. They solve equations of the form $x^n = x$, $n \neq 2$, equations which thus differ with the law of linear logic $x^2 = x$.

---

*[See PO & LC, p. 16]

That is, $i^2 \neq i$, $\varepsilon^2 \neq \varepsilon$, $w^2 \neq w$, and $p^2 \neq p$. In geometrical interpretation, each of these hypernumber unities is, like real unity, r, represented by a unit length along a distinct number line, a number-dimension perpendicular to the axis of r and each of the other hypernumber axes. The i and r axes together form the well-known "Complex plane" or "Argand diagram". Hypernumber space is a higher-dimensional generalization of this. It is not "hyperspace", meaning "$\mathbb{R}^n$", $n > 3$, the crossing of four or more mutually perpendicular r axes. Hypernumber axes are not r numberlines. Dr. Charles Musès, the principal discoverer of the hypernumbers, calls it "metaspace", the space of metadimensions.[51] Restricting to three axes at a time (which is all we can depict with any facility), hypernumber metaspace looks like this:



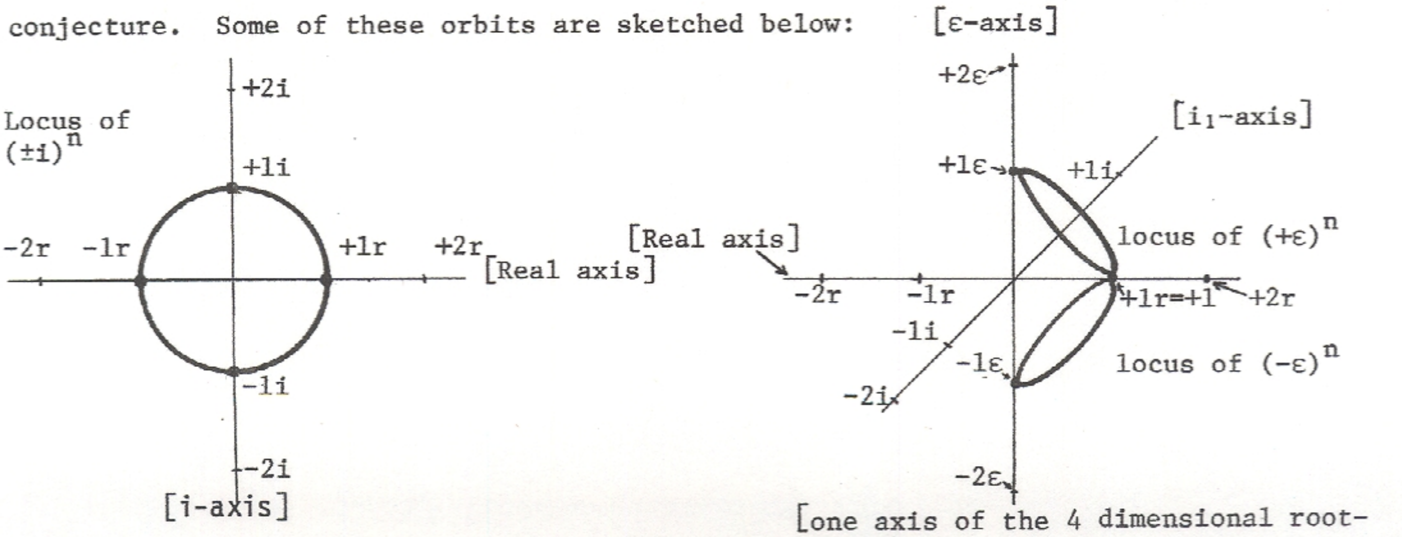
[Excerpt] [Excerpt]

Figure 2: View of two different 3-dimensional "sides" of Hypernumber Space [Also known as "The Space of Metadimensions" or "Metaspace"]

Each hypernumber unit operator has thus its own distinct unit interval, with its own unique direction or dimension and its own unique "hyper-Boolean" logic. The arithmetic properties of the unit interval of 0 and i form one such "logic". The unit intervals [0,ε], [0,w], and [0,p] form others. The "logic" of the ε operation is governed by the equation $x^3 = x$, the logic of the w "verb" by $x^7 = x$, the logic of i by $x^5 = x$, and so on. It is natural, at this point, to wonder if these nonlinear equations express the "fundamental laws" of a new nonlinear logic, a "logic of paradox" which transcends classical Aristotelian logic. It is natural to wonder if these nonlinear logical laws capture the "logic" of nonlinear differential equations. The solutions of linear differential equations are analogues of the solutions of Boolean linear logical equations. The latter solutions always belong to the unit interval of Real Unity: [0,1]. Perhaps, then, the solutions to nonlinear differential equations are analogous to the solutions of nonlinear logical equations, the paradoxical ("uninterpretable") Boolean equations $x^n = x$, $n > 2$. Perhaps, that is, the solutions to nonlinear differential equations involve hypernumbers!

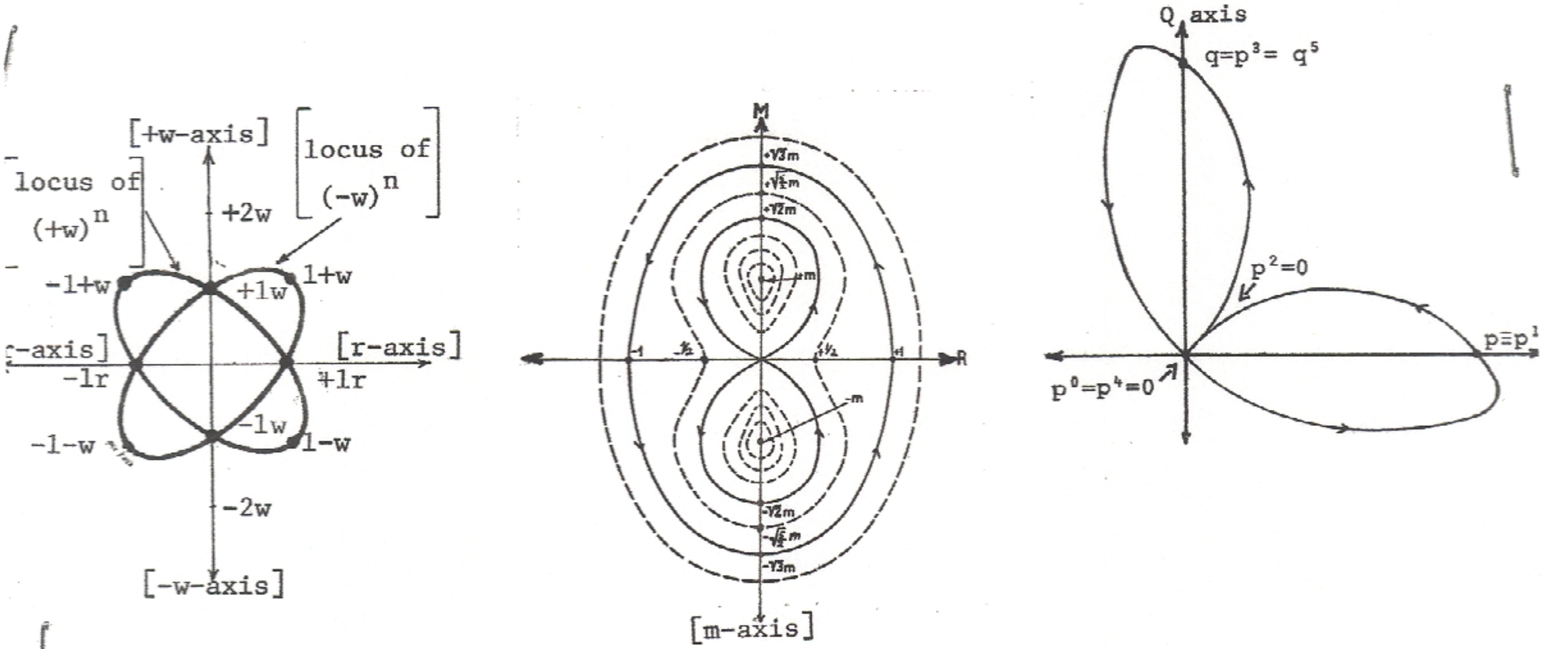
Power-Orbits and Circular Logics

Observation of the power-orbits of the eight varieties of hypernumbers which have them -- i, ε, w, p, q, m, Ω, and v -- provides strong support to the above conjecture. Some of these orbits are sketched below:



[one axis of the 4 dimensional root-space (power-space) of ε is omitted: the i_o-axis]

Figure 3: Selected Hypernumber Power-Orbits


[+w-axis]
locus of (+w)^n
locus of (-w)^n
+2w
-1+w
1+w
+1w
[r-axis]
r-axis]
-1r
+1r
-1w
-1-w
1-w
-2w
[-w-axis]
M
+√3m
+√(3/2)m
+√2m
+m
-1
-½
+½
+1
R
-m
-√2m
-√(3/2)m
-√3m
[m-axis]
Q axis
q=p³= q⁵
p²=0
p=p¹
p⁰=p⁴=0

The power-orbits are the loci of u^n where u is a hypernumber unity and n ranges over the Real numbers. This means that n takes on fractional, irrational and negative as well as positive integral values. The power-orbit of a number is the locus of all its Real powers. It therefore includes its nth roots [(1/n)th powers]. Even more, it includes the locus of irrational powers. The power-orbit is thus a power-continuum, appropriately represented by the solid-line curves above.

The above is the technical definition. The meaning and significance of the power-orbit is best grasped in the following terms. For any (hyper)number u, the power-orbit of u characterizes the operation u denotes. That orbit is "characteristic" of u in the same sense in which the operating curve of a circuit element (circuit operator) is called the "characteristic" of that device. The power-orbit tests the operator u upon itself. It "ferrets" the (operational) "field" of u⊥r space by using u itself as the "test particle". Dr. Musès, originator of the concept, defines the power-orbit, or "unit power field", thusly:

> The metadimensions are isomorphic to the kinds of number, which in turn are characterized by their unit power fields [or "power-orbits"], i.e. the function their unit traces out in a suitably defined representation plane [or in some cases a higher-than-two-dimensional representation space] when it is reiteratedly multiplied by itself. Thus the unit field [power-orbit] form shows the self-reflexive operation of the given unit or kind of number.[52]

That is, the power-orbit is the locus of a special type of "reflexion" or "reflexive function". The power-orbit is the trajectory generated by the continuous self-operation of the hypernumber unit; by the endlessly recurrent self-interaction of the process that unit denotes. It is the "path of self-multiplication" of the number, "multiplication" here understood in its generalized sense as signified by operator (self-) juxtaposition: $u^n \equiv \underbrace{uuu\ldots u}_{n}$.

The operational "logic" of the hypernumbers i, ε, w, p, q, and m depicted in Figure 3 is thus literally circular, in contrast to the rectilinear logic of the multiple ℝ-numbers, and the "fixed-point", "static equilibrium" logic of r itself. Though only two of the power-orbits are exactly circular, all are "circular" in the sense of being closed, self-re-entering paths.

The Fundamental Theorem of Nonlinear Algebra

Adding the eight newly discovered kinds of number ε, w, p, q, σ, m, Ω, and v to the previously known numerical calculus of r and i, expands arithmetic--and therefore algebra--considerably. The resulting "hyperarithmetic" and "hyperalgebra" exhibit extraordinary properties. In particular, hyperalgebra is a nonlinear algebra.

To date, linear algebras have been the exclusive focus of mathematical study. The concept of a nonlinear algebra has scarcely been considered. The reasons are not hard to find. A nonlinear algebra implies a nonlinear *arithmetic*. A nonlinear arithmetic implies *nonlinear numbers*. Until the discovery of the hypernumbers, in conjunction with the generalization of the operator concept, which they suggest, the very meaningfulness of such concepts was in doubt.

To understand the nature of a "nonlinear number", we must look at the definition of a nonlinear *operator*:

> An operator J is said to be linear if the effect of operating on the sum of two entities, e.g. functions, is equal to the sum of the effects of operating on them separately, i.e.
>
> $$J(f + g) = J(f) + J(g) \quad \text{and} \quad J(af) = aJ(f)$$
>
> where *a* is a constant. J is *nonlinear* if it is not linear.[53]

The property $J(f + g) = Jf + Jg$ is usually called "superposition" and $J(af) = aJf$ "homogeneity". But suppose, in light of our generalized "operator" conception, we look upon a, f, g and especially J *all* as (generalized) "numbers", hypernumbers. Then we immediately recognize the first property as the "distributive law", and the second as the "commutative law" (*a* commuting with J). *Nonlinear numbers would evidently be nondistributive and noncommutative numbers*!

It turns out that hypernumber arithmetic, *in general*, is noncommutative, nondistributive, and nonassociative as well! "In general" means that there exist cases of hypernumber computations which violate each of these laws. A case in point is

$$0 = w(w-w) = w(w+(-w)) \neq w \cdot w + w \cdot (-w) = 2w$$

Recall that $ww \equiv w-1$. The product $w(w-w) = w \cdot 0$ is, of course, zero if the subtraction is performed first. But $w(-w) = (-w)^4(-w) = (-w)^5 = w + 1$. Therefore $ww + w(-w) = w-1 + w+1 = 2w \neq 0$.[54] Thus the hypernumber w is a nondistributive operator, hence a nonlinear number!

However, nondistributive behavior is a special case in w-arithmetic, which Dr. Musès classifies as an "almost distributive" arithmetic. This behavior has intimately to do with the ambiguity introduced by the dual nature of the w power-orbit, one ellipse for +w, another for -w. It arises from the circumstance that both factors in the crucial product $w\cdot(-w)$ occur as values on each power-orbit, i.e. the points +w and -w are crossing points of the two orbits. However, we are informed that:

> ...hypernumber arithmetics in general are fully nondistributive if they have power-orbits that are not confined to points, straight lines, circles, or ellipses--in which event similarly shaped and oriented orbits no longer ensure distribution.[55]

The power-orbits of the "twin" hypernumbers p and q, as well as those of m and Ω, qualify.

Noncommutativity arises in many hypernumber contexts. For example:

$$1+w = w(-w) \neq -w(w) = 1-w$$

and

$$\varepsilon i = -i\varepsilon$$

So the hypernumbers in general are nonlinear numbers, producing a nonlinear arithmetic. A nonlinear arithmetic means a nonlinear algebra.

The linear algebra of Real numbers plus "Imaginary" numbers, i.e. of the "Complex" numbers, exhibits an important kind of "completeness". This "completeness" is codified in the "Fundamental Theorem of Algebra". The theorem states that Complex algebra is "complete" in the sense of solvability; is "closed" for solutions. Every equation of Complex algebra, no matter how large its (Complex) coefficients or degree, has solutions in the space of Complex numbers. Real algebra is not complete in this way. Equations like $x^2+1 = 0$ have no solutions in Real space.

A conjecture by Musès states there can be no more than eight basic kinds of number beyond i. This implies that the eight hypernumber varieties ε, w, p, q, σ, m, Ω, and v already isolated are comprehensive. No further fundamental operators of this sort remain to be discovered. The argument of the conjecture runs as follows. Eight basic rules define Real arithmetic, including the commutative, associative, and distributive laws. Each of the eight post-i hypernumbers abrogates one or more of these eight rules. This abrogation(s) characterizes that hypernumber "species". Together, these eight varieties form a pattern of abrogation which leaves no room for another viable number-variety. And after all eight rules have been abrogated, nothing remains to *distinguish* further kinds of numbers. Attempts at isolating additional hypernumbers yield operators *indistinguishable* from one or another of the species already identified.[†]

If the ten+dimensional[*] number-space formed by orthogonally crossing the ten plus number-axes or"number-dimensions" $r \perp i \perp \varepsilon \perp w \perp p \perp q \perp \sigma r \perp m \perp \Omega \perp v$ is thus complete, we might wonder if its algebra gives rise to a theorem analogous to the "Fundamental Theorem" of Real-Imaginary or "Linear" algebra. That is, does the "Fundamental Theorem" of $i \perp r$ algebra *generalize* to a "Fundamental Theorem of *Nonlinear* Algebra"? This theorem would mean that this ten-dimensional Generalized (Hyper) Number-Space or Operator-Space is "complete" for solutions. That is, the theorem would hold that all equations of "hyperalgebra" have solutions within this space; that is, solutions expressible in terms of the ten basic operators. No equation would lead outside this space.* We conjecture that such a theorem indeed holds true.

---

[*] [Here we assume that certain cases of "compound" hypernumbers which, like iw = wi, create additional axes beyond the basic ten, are included where needed, for they do not represent new *basic* hypernumbers, being expressible, rather, in terms of the basic ten. Subvarieties of the basic hypernumber varieties also add axes to the space, making many more than ten dimensions.]

[†] [See Section IV.(Appendix), *Basic Papers on Hypernumbers*: "Hypernumbers II" by Dr. Charles Musès, pp. 63-64.]

The exciting thing about this conjectured theorem is that, if true, it would guarantee the expressibility, in terms of hypernumbers, of the solutions to nonlinear *integro-differential* equations. Our suspicion that hypernumbers and their "logic" are involved in these solutions would be borne out.

Why would this be so? This would be so because the hypernumber σ is none other than the *integration*[†] operator of calculus. The *differential*[†] operator is σ^{-1}, the first negative power of sigma. Thus integrodifferential equations, including the *nonlinear* variety--the most numerous--fall within *algebra* in this expanded sense. And if that algebra's arithmetic base can supply solutions to all of its possible equations, then explicit analytical solutions to nonlinear integrodifferential equations can be had in closed form in terms of *hypernumber functions*.[*]

Dr. Musès seems to have suspected this when he wrote, in the founding paper on nondistributive algebras and hypernumbers:

> There remain two open vistas for the development of the theory of functions: (1) a deepening of number theory, hence of algebra and function theory, in the direction of more inclusive kinds of numbers; and (2) a development of a theory of nonlinear operators and nonlinear differential equations. These two paths need not be unrelated.[56]

But what might these hypernumber solving-functions look like?

The Logic of Limit Cycles

Below are free hand renderings exemplifying *limit-cycles*--the limit-cycle being one of the most common solution-geometries for nonlinear integrodifferential equations in the state-space depiction of their solutions.

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[*] [See PO & LC, p. 50, quote with footnote number 37.]

[†] [with respect to *time*, so that partial differentials and integrals must be constructed by time-parametrizing the other variables, i.e. by making them values of parametric functions of t. Musès defines sigma as follows:

$$\sigma^n(\ ) \equiv \underbrace{\int\int\int\ldots\int}_{n} (\ )\cdot s(t)dt$$ where s(t) is the Heaviside unit step function,

and $\sigma^{-1}(\ ) \equiv \frac{d}{dt}(\ )s(t)$, the Heaviside Operational Calculus form of the differential operator.]

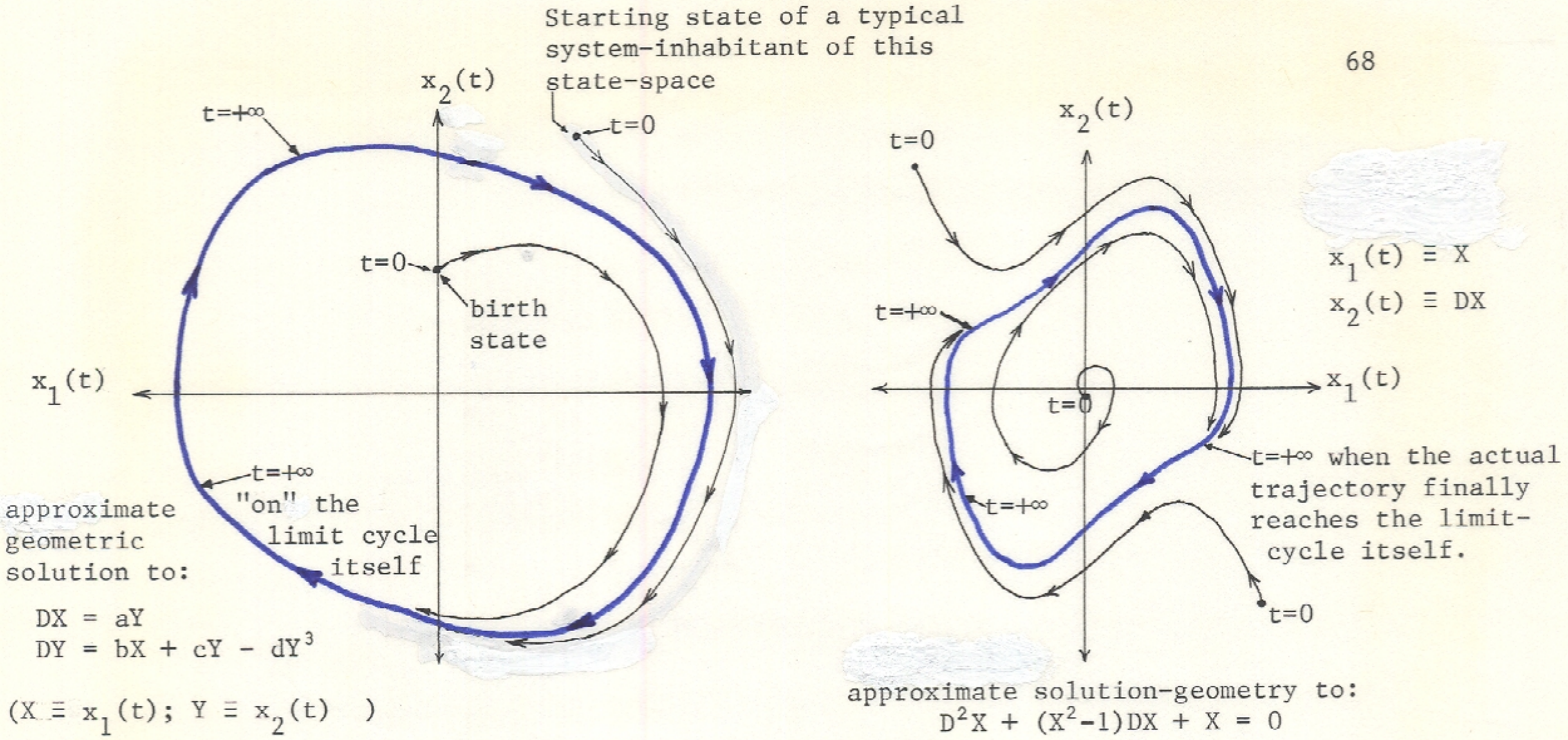
Starting state of a typical system-inhabitant of this state-space
$x_2(t)$
t=+∞
t=0
t=0
birth state
$x_1(t)$
t=+∞ "on" the limit cycle itself
approximate geometric solution to:
$DX = aY$
$DY = bX + cY - dY^3$
$(X \equiv x_1(t);\ Y \equiv x_2(t)\ )$
t=0
$x_2(t)$
$x_1(t) \equiv X$
$x_2(t) \equiv DX$
t=+∞
$x_1(t)$
t=0
t=+∞ when the actual trajectory finally reaches the limit-cycle itself.
t=+∞
t=0
approximate solution-geometry to:
$D^2X + (X^2-1)DX + X = 0$

Figure 4: Typical Limit-Cycle Solutions

Now notice that these look distinctly like modified versions of some of the power-orbits, irregularized in shape in some cases, but "take-offs" upon them nonetheless--"variations" on their closed-orbital "theme".

Could the limit-cycles possibly be power-orbits? That is, could the reflexive functions which generate the power-orbits, of form u^n, constitute the core of a new kind of function which is capable of expressing limit-cycles in explicit analytical form? If so, for one thing, u^n would have to be transformed into a function of time. As it stands, it describes only a static locus.

But this is easy. We simply exchange the n for a t, obtaining

$$\underline{x}(t) = u^t$$

as the temporalized power-orbit formula. We define the "principal value" of u^t, for cases of fractional values of t and multiple roots of u, so that the root selected is consistent with a continuous trajectory for u^t. Then, in the typical case, we get a "state-vector valued function" for two* state-variables, call them $x_1(t)$ and $x_2(t)$, with u_j and u_k acting as "unit-(state-)vectors":

$$\vec{x}(t) \equiv \begin{bmatrix} x_1(t)\cdot u_j \\ x_2(t)\cdot u_k \end{bmatrix} \equiv u_k^t = x_1(t)\cdot u_j + x_2(t)\cdot u_k$$

wherein u_k is a unit hypernumber and, usually, $u_j = r$.

or, to take the only example hitherto familiar; that of $u_k = i$ ($\rightarrow u_j = r$):

$$\vec{x}(t) = \begin{bmatrix} x_1(t)r \\ x_2(t)i \end{bmatrix} = i^t \equiv e^{i\frac{\pi}{2}t} = x_1(t)r + x_2(t)i$$

$$= (\cos \tfrac{\pi}{2}\, t)r + (\sin \tfrac{\pi}{2}\, t)i$$

Such functions describe closed trajectories in the $x_1(t)u_j \perp x_2(t)u_k$, or $u_j \perp u_k$, state-space.

However, such closed trajectories are not yet limit cycles. In a true (attracting) limit-cycle solution, the closed path of the limit-cycle itself is never reached by the actual system trajectories in all finite time. The actual trajectories merely approach this orbit asymptotically. Also, i^t describes a perfectly circular orbit, whereas we want, in general, an irregular "variation" on the (in this case) circular theme of the pure power-orbit.

To accommodate the "transient" (decaying) asymptotic process, as well as the permanent "variation" modifier, we introduce a "coefficient-function" g(t). The power-orbit "core" u_k^t expresses the "essence", the "genotype" of the non-linear system, whereas g(t) expresses the effect of external† circumstances which

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*[There are exceptions. For example, the power-orbit of ε_1 is a circle oriented in the four-dimensional space $r \perp \varepsilon_1 \perp i_1 \perp i_0$, where i_1 and i_0 are two distinct forms of $\sqrt{-1} = \sqrt{-r}$. The ε_n power-orbit (temporalized) is given by:

$$\varepsilon_n^t = \cos^2(\tfrac{\pi}{2}\, t)r + \sin^2(\tfrac{\pi}{2}\, t)\varepsilon_n - \tfrac{1}{2}\sin(\pi t)i_n + \tfrac{1}{2}\sin(\pi t)i_0 .]$$

†[That is, g(t) represents the independent action of the environment of the system upon its present state, at time t, ∀t, whereas the "core" function, $u_k^{f(t)}$, expresses the effects of the past activity of the system itself upon its present state.]

block expression of the essential behavior, producing a distortion and deviation from essence, a "phenotype".

To handle other commonly observed effects, such as speeding up (or slowing down) of the state-space velocity*, we generalize the t exponent to an "exponent-function" f(t). Thus we provisionally propose the following as the general form of hypernumber reflexive function solutions to nonlinear integrodifferential equations of the limit-cycle variety:

$$\vec{x}(t) = g(t)u_k^{f(t)} \qquad (*)$$

wherein u_k is a unit hypernumber operator, such as ±i, and wherein g(t) is factorable

$$g(t) = a(t)v(t) \text{ [where (normally) } a(t)\to 1 \text{ as } t\to\infty]$$

such that a(t) controls the asymptotic behavior of the trajectories, and v(t) accounts for any permanent distortion or variation of the shape of the ultimate limit-cycle proper from the pure form of the power-orbit, given by u_k^t. The function g(t) controls the ever-changing amplitude of the limit-cycle vibration, whereas f(t) controls its ever-changing frequency. Notice that, in the special case where u_k is not a hypernumber, that is, if $u_k = r \equiv +1$, then the form (*) reduces to a Real-valued-function:

$$g(t)\cdot 1^{f(t)} = g(t)\cdot 1 = g(t)$$

where g(t) is the known Real-valued solution to a linear differential equation. This means, for one thing, that linear systems are "passive"; largely determined by their environment; externally determined rather than self-determined.†

It is conceivable that reflexive functions of compound hypernumbers in the form of sums of two or more different hypernumber units, may provide explicit solutions for "Strange Attractor" nonlinear systems as well.

What remains to be determined to settle this solution-conjecture? Principally, the exact "rules of formation" of the proposed hypernumber reflexive function solutions (hereafter called "hyperfunctions" or "metafunctions" for short). Such rules would have to tell us, given the general features of the differential equation to be solved, including coefficients, degree, and order: (i) which specific one of the basic hypernumber unities would have to play the u_k role in the solution; (ii) the exact form of the function f(t); and (iii) the exact form of the function g(t).

---

*[the velocity of the system-point along the state-space trajectory]

†[see PO & LC, p. 19.]

At present, with these rules of formation still unknown for general cases, we seem to have exchanged one unknown function, $\vec{x}(t)$, for two: f(t) and g(t). Still the attainment of a working hypothesis as to the explicit form of the unknown nonlinear solution-functions, $\vec{x}(t)$, is a significant gain.

Specific examples of functions belonging to the form (*) have been plotted, yielding graphs mimicking persuasively the attributes of limit-cycles attractor trajectories. But direct differentiation of mimicking functions tried out so far yields no well-known examples of nonlinear differential equations. The exact form of the asymptotizing factor a(t) is still in doubt.

At this juncture, it should be noted that the state-space implied and constructed by such hyperfunctions is anisotropic.* These functions take subspaces of metaspace, hypernumber space, to be the state-space, whereas heretofore state-space was always presumed to be a special case of "hyperspace", $\mathbb{R}^n$. Hypernumber state space is anisotropic in the same sense that the Complex plane, I x $\mathbb{R}$ or $\mathbb{C}^1$, is anisotropic. The "directionally differentiated" character of $\mathbb{C}^1$ owes to the fact that the units of the two axes, i and r, differ in operational quality. Hyperfunctions and the hypernumber state-space they imply assign to each of the different state-variables, i.e. the several measurements of different system qualities, each with its distinctive unit of measure, a qualitatively different unit (state-)"vector", in the form of a different hypernumber unit, and associated axis or dimension. The possible conceptual meaning and importance of this anisotropy will emerge under the next heading.

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*[See PO & LC, pp. 31; 79; 89]

Here, we note only that anisotropic state-spaces and hypernumber functions are not confined strictly to the nonlinear domain [assuming this proposed solution proves out], with Real functions and Real state-space confined to exclusive comprehension of solutions in the domain of linear differential equations. The anisotropic hypernumber state-space $I \times \mathbb{R}$ or $\mathbb{C}^1$ is already heavily implicated in the solution to the general linear system. The general term of the general linear solution-function can be written in the form

$$Kt^k e^{(a+ib)t} \qquad (**)^{57}$$

where, in special cases, k and b or a can be zero, thus reducing to the pure "Real" or pure "Imaginary" cases respectively. This function (**) contains the temporalized $\mathbb{C}^1$ power-orbit, as can be seen by the following chain of equivalences* (involving the Euler identity $i \equiv e^{i\frac{\pi}{2}}$):

$$Kt^k e^{(a+ib)t} = Kt^k e^{at} e^{ibt} = Kt^k e^{at}[\cos(bt)r + \sin(bt)i]$$

$$\equiv Kt^k e^{at} \cdot e^{\frac{\pi}{2}\cdot ibt \cdot \frac{2}{\pi}} = Kt^k e^{at}(e^{i\pi/2})^{\frac{2}{\pi}\cdot bt} \equiv Kt^k e^{at} \cdot i^{\frac{2}{\pi}\cdot bt}$$ [using our principal-value convention]

Though it can describe a closed orbit (if k=a=0), this function does not describe a true limit-cycle, because it lacks any asymptotic factor, and the closed orbits produced lack the character of "attractors"†. Nevertheless, it comes under the form (*)

$$\vec{x}(t) = g(t)i^{f(t)} = x_1(t)r + ix_2(t)$$

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*[see PO & LC, pp. 95; 97-98]

†[or of "repellors" either. The resulting closed or "self-reentering" trajectories are termed "neutral" or "neutrally stable".]

with[†] $g(t) = Kt^k e^{at}$; $f(t) = \frac{2}{\pi} \cdot bt$. If $a>0$, this hyperfunction determines a trajectory outspiraling from the origin in a C^1 state-space. If $a<0$, an inspiraling solution results. If $a=0=k$, the pure "Imaginary" case ensues, and a closed, circular [or, in general, *elliptical*] trajectory is generated.

True, the usual practice is to squelch this Complex solution, and force the function back into a "Real" state space, by separating the "Real" and "Imaginary" parts of (**) and treating each as a separate, linearly independent solution. Therefore, with loss of unity, the general term of the general linear solution is traditionally described as a linear combination of the "two" function forms[58] (with C,D = constants):

$$Ct^k \cdot e^{at} \cdot \cos(bt) \quad \text{and} \quad Dt^k \cdot e^{at} \cdot \sin(bt)$$

instead of the single function (with $K = rk_1 + ik_2$):

$$Kt^k \cdot e^{at} \cdot [\cos(bt)r + \sin(bt)i] = Kt^k \cdot e^{at} \cdot i^{\frac{2}{\pi}bt}$$

Thus hypernumber reflexive functions and spaces have important and useful applications even in the linear domain. And i is not the only hypernumber that comes up in the solution of linear differential equations. For example, ε does also,[*] though this is hidden in current treatments, since ε is not yet generally recognized, unlike i. In fact, the most important applications of hypernumber methods hitherto have occurred in the domain of *linear* algebra, including the astounding discovery of *nonassociative* cases of matrix multiplication, of square roots and other fractional powers of matrices, and, most startling of all, the discovery of *expressible inverses* for singular matrices.[58.5]

In light of this form of the general component of linear solutions, involving powers of the important Real number* e, we can restate our solution-proposal. Besides the power-orbit u^n, hypernumbers have another characterizing trajectory, called by Musès the *exponential orbit*[59], given by $e^{u\theta n}$ where, as before, u is the hypernumber unit in question, and n ranges over the Reals, and where θ is a Real constant characteristic for each particular u. For i, $\theta = \frac{\pi}{2}$ and the power orbits and exponential orbit *coincide*. Since $i = e^{i\cdot\frac{\pi}{2}}$, $i^n = e^{\frac{i\cdot\pi\cdot n}{2}}$ for every n. To u=r corresponds $\theta = 1$ and $e^{rrn} = e^{rn} = e^n$, an "open" orbit. For hypernumbers u other than i, the power-orbit and exponential orbit are distinct. But the exponential orbit may still be a *closed* orbit. It may differ in shape from that of the power-orbit, or it may be a version of the same shape differently deployed.

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[*] [See PO & LC, Section 3, "Power Orbits", subsection 1, "Hypernumbers": "Applications to Linear Algebra".]

[†] [When $k \leq 0 > a$, then $(t^k e^{at})$ forms the asymptotic factor A(t), decaying to 0 as $t \to \infty$, while the Complex constant K forms the variation factor V(t).]

For closed orbit cases of exponential orbits, our proposal for explicit hypernumber solutions to nonlinear differential equations can be reformulated as follows. We seek a *straightforward extension* of the general linear solution basis $Kt^k e^{(a+ib)t}$, an extension to the form (where $G = rk_1 + uk_2$)

$$Gt^k e^{(a+ub)t} \quad \text{or, more generally, } g(t)e^{u\theta f(t)} \qquad (***)$$

wherein u is a hypernumber unit *other than i*, or a multiple and/or compound hypernumber possibly involving, but not limited to, i multiples and compounds.

All of our proposed solution-functions formulate *self-interaction* of hypernumbers u. They do so either *directly*, as a recurrent "reflexion" of u itself, or *indirectly*, via the self-operation of $e^{u\theta}$. And they formulate this self-interaction *as a function of time*. Their root forms are $s(t) = u^t$ and $s(t) = e^{u\theta t}$. Both of these root forms may provide the previously missing core functions necessary to express limit-cycle solutions, and perhaps strange attractor solutions as well, in a new "closed form". Such functions of time are *temporalized hypernumber reflexive functions* or, as we shall call them hereafter, *dynamical* reflexive functions.

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*[See PO & LC, pp. 40-41; 76. The irrational Real number e might be characterized as the "organic growth operator".[60]]

We conclude this exposition of the hypothesized solution to the problem of nonlinearity with the following observation. The metafunction form (*) also provides a solution to the "stability of denotation" and "explicit timing" problems of the "process algebra" interpretation of generalized arithmetic (see p. 30, below).

If we consider that the "spontaneous transition" of *a* into *e*, then of *e* into *g*, and *g* into *k*, and so on results from the self-interaction of *a*, then we want the state of the system-eventity whose *original* state (say time t=0) was *a*, to be given by a hyperfunction called s(t) such that

$$s(t) = a^{f(t)} = a^{2^t}$$

in which case (see pp. 38-39 below):

$$t = 0 \text{ implies } s(t) = a^{2^0} = a^1 = a$$

$$t = 1 \quad \rightarrow \quad s(t) = a^{2^1} = a^2 = a(a) \equiv e$$

$$t = 2 \quad \rightarrow \quad s(t) = a^{2^2} = a^4 = a^2(a^2) \equiv g$$

$$t = 3 \quad \rightarrow \quad s(t) = a^{2^3} = a^8 = a^4(a^4) \equiv k$$

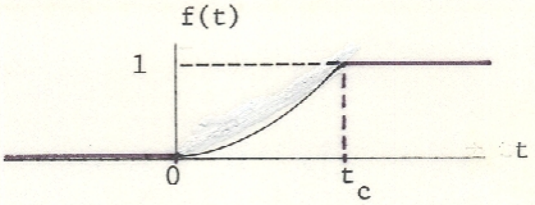
and so on

The "exponent-function" $f(t) = 2^t$ gives the explicit timing of the process in which initial state *a* successively "mutates" to other states *as a result of its self-interaction*. Thus the problems of explicit timing and denotational consistency are resolved in a single stroke. For "mere flexions" (see PO & LC, pp. 15, 18), as opposed to "*re*flexions" or "reflexive functions", the same method of "explicit temporalization" applies

$$ab \to c \text{ wherein } a \neq b \text{ becomes } a^t b \Big|_{t=0}^{t=1} \to c \quad \text{or } a^{f(t)} b \to c$$

with the constraint on $f(t)$ that $f(0) = 0$ and, for some time t_c at which the product c is fully produced, and for all later times $t > t_c$, $f(t_c) = 1$, while for t inbetween: $0 < t < t_c$, we have $f(t)$ such that $0 < f(t) < 1$ and monotonically increasing throughout the interval (0,1). That is, f would be some form of "unit ramp function::

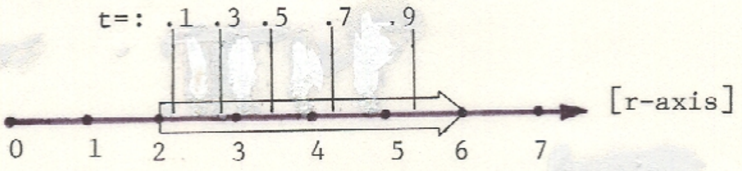
Figure 5: General Form of the Exponent-Function for "Mere" *Interactions* [See PO & LC, pp. 15, 18.]



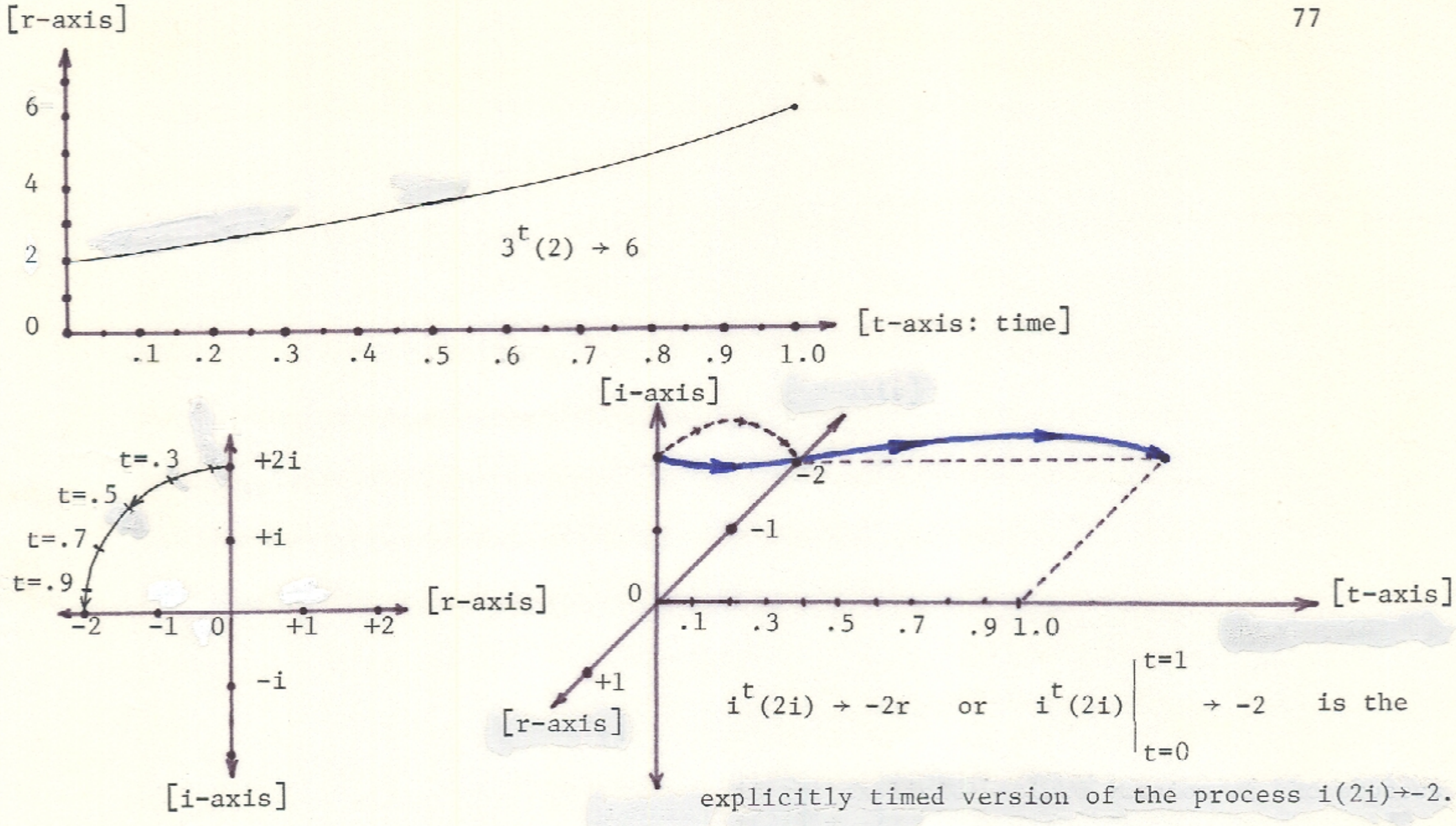
In this way $s(t)=a^{f(t)} \cdot b$ will describe the operational action of a upon b as a movement in number-space, starting at the number(point) b at $t = 0$, and moving steadily toward the number-point c as $t \to t_c$, but stopping forever at c once c is reached. That is, $s(t)$ will describe the interaction

$$ab \to c$$

with explicit timing. It will describe the *calculative process* $ab \to c$ as a timed movement, a *dynamical trajectory* in the number-space containing a, b and c. Here a, b and c can be regarded as denoting "states", or *points* in a state space. That is, their number-space can be interpreted as a state-space, and the calculative expression $ab \to c$ as a state-transition.



$3^t(2) \to 6$ or $3^t(2) \Big|_{t=0}^{t=1} \to 6$ is the explicitly timed version of the process $3(2) \to 6$.



$i^t(2i) \rightarrow -2r$ or $i^t(2i)\Big|_{t=0}^{t=1} \rightarrow -2$ is the explicitly timed version of the process $i(2i) \rightarrow -2$.

[Figure 6: Explicit Timing of Number-Movements cf. PO & LC, p. 15]

It should always be possible to rescale t so that the form $a^{f(t)} \cdot b \rightarrow c$ can be reformulated in the simpler form $a^t b \rightarrow c$ with the restriction $t \varepsilon [0,1]$.*

If these proposed solutions work out, then, indeed, the 'Hyper Boolean Logical Laws' $x^n = x$, $n > 2$ will turn out to have captured the "logic" of nonlinearity, the "logic" of nonlinear differential equations, just as $x^2 = x$ captured the logic of linearity, of linear differential equation dynamics. The unit-interval arithmetics and algebras of the hypernumbers will then turn out to provide models for the nonequilibrium logic of sustained oscillation, the logic of limit-cycles (and perhaps of "strange attractors" as well), just as the Boolean algebra and arithmetic of Real unity (r or +1) provided models for "first degree" or "linear" logic and for the equilibrium or "fixed point" solutions of linear differential equations. The operational properties of these hypernumber unities will then make clearly visible the long felt connections between the logic of paradox, circular logics, and the logic of reflexive processes or of self-interaction--the logic of reflexion.

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*(Here "ε" denotes the relation of set-inclusion, not the hypernumber ε)

Throughout the third and final section of this paper, and the remaining subsections of this second section, we will explore the conceptual meaning, the theoretical implications, and the descriptive applications of the hyperfunction forms (*) and (***) for mathematical modeling of economic reproduction. We will be on the lookout especially for versions of the economic reproduction model capable of encompassing technical mutation or innovation, credit processes, and cyclical trends, and thus capable of formulating our hypothesis.

The Ideographic "Leverage Principle"

In the development of written language from ancient times, the ideographic and pictographic principles and tendencies were increasingly eclipsed by the phonogramic or phonetic principle and tendency. Only in mathematics did the ideographic principle sustain a vigorous growth into the modern era (with the notable, partial exception of written Chinese). The causes of phonetic dominance are clear. They stem from the vast reduction of mnemonic requirements made possible by what I shall call the 'Phonetic Leverage Principle'. By the "leverage" of a symbolic system I mean a measure of the "versatility" of its basic symbols. The greater the multiplicity of "objects" or processes in the representation of which a given basic symbol can be used, the greater the "leverage" of that symbol, and the greater the leverage of the system employing it.

An ideographic script must have a different symbol for every idea considered distinct. Suppose such a script arises late in an advanced society wherein the long historical labor of philosophy has achieved a considerable conceptual unification. Suppose that most concepts can be expressed by suitable combinations of the symbols for a small group of universal concepts considered to be the fundamental ground from which all other ideas grow. Then this ideography might hope to approach the "symbolic leverage" and mnemonic economy of a phonetic script. This assumes, of course, that all the new concepts that are continually arising can be symbolized in terms of this ground; that none of them represent a revolutionary conceptual innovation obsoleting, and forcing, in fact, a revision of, that "ground". The history of ideas demonstrates the questionable character of any such assumption. Lacking such conceptual unification, with new ideas demanding new symbols all the time, and with a large coterie of symbols from the start, an ideographic script quickly becomes mnemonically unmanageable.

Likewise, if a pictographic writing system could be based on a profound and exhaustive classification of the forms or "gestalts" available to visual perception, and be "coded" based upon such a classification, then a workable system of picture-writing, though rather abstract, might be conceivable. Even if the visual forms of natural scenes could be comprehensively codified, such a pictography would have to cope with the new "shapes" of the new kinds of objects constantly being added to the world by human ingenuity (and perhaps by novel occurrences in extra-human nature as well). But lacking such a "taxonomy of visual impressions", a pictographic script, in a developing culture, quickly becomes as mnemonically burdensome as any ad hoc ideography.

Classificatory processes like the above might make ideographies and pictographies workable as universal, general purpose written languages. But such prowess requires a heritage of profound scientific knowledge--both of the world and of human faculties--of a degree which does not arise in the early stages of civilization and which, in fact, has still to be fully achieved in our own culture. The phonetic leverage principle, on the contrary, requires only a deep observation of the sounds of utterance employed by a given language community. The key to the power of the phonogramic symbol is its indirect, circuitous approach to symbolization. The phonogram is a symbol of a symbol. Phonetic script is a representation of a representation, a "representation of the second degree". Directly, the phonetic symbol represents a sound of speech. That sound is a component of a larger sound, itself already a symbol, a phonic or vocal symbol for a meaning: a word. This stratagem enables a 26 member alphabet to encompass an ever-expanding, living language. The 26 letters are sufficient to formulate an endless variety of new ideas and new "objects", never envisaged by the originators of the script. An indefinite number of new words--nouns, verbs, etc.--can find symbolization in terms of this same, finite spectrum of phonograms. The "nonlinear" ("second degree") strategy of symbolization achieves a truly open-ended symbolic system with a potentially infinite symbolic leverage. Two symbolization levels achieve what one couldn't.

No workable ideography of process, no practicable "process algebra", is conceivable without an analogous provision for mnemonic economy, symbolic leverage, and evolutionary open-endedness.

The hypernumber operators, and the state-space interpretation of their space, appear to provide this.

The diagram below indicates the manner of this provision, and the analogy with phonetic leverage. The ideographic leverage principle, like the phonetic, involves the "second degree" of representation, a "symbolization of a symbolization". [The curved arrow sign stands for the word "symbolizes", and is directed from the thing symbolizing to the thing symbolized]:

LEVELS OF SYMBOLIZATION:

PROCESS ← state-space TRAJECTORY-type ← hypernumber(operator ideogram)

(system) (eventity) | (in hypernumber state-space: $r \perp i \perp \varepsilon \perp w \perp p \perp q \perp \sigma r \perp m \perp \Omega \perp v$) | ("system-verb")

MEANING ← SOUNDS of utterance ← phonograms; letters of the phonetic alphabet

(idea, external process) | (vocal, phonic symbol--morpheme, or word)

In the phonetic system of representation, sounds of speech form the primary layer of symbolization. This layer mediates the interface of the ultimate signifiers, the phonogramic characters, with the ultimate meanings, the ideas and processes referred to by groupings of literal symbols, words. In the ideographic system of hypernumber arithmetic, trajectories, that is, patterns of movement through hypernumber space, constitute the immediate layer of symbolization. The hypernumber formulas, flexive and reflexive functions of time, do not symbolize the systems or processes being modeled directly. The hypernumber functions directly describe the trajectories. Thereby, indirectly, they describe the processes. They get at the processes by way of the trajectories, just as phonetic symbolism gets at the processes to be described by way of speech-sounds, and not directly.

The state-space interpretation of the space of hypernumbers is accomplished as follows. The key, state-revealing measurements about the system (i.e. process, eventity) to be modeled are designated. These measurements will be the "state-variables" of the system. This designation is made in accord with the purpose of the modeling. The state-space representation of a given system may differ for different purposes. Often a selection of as few as three state-measurements out of the innumerable possibilities will suffice, and is convenient for graphic purposes. Each such measurement or state-variable is then identified with the appropriate axis in hypernumber space. That is, each measurement is assigned to the axis of a definite hypernumber. That axis becomes the scale of that state-variable. The unit of measure of the state-variable is identified with the (hypernumber) unity of that axis, whether that unit be, for example, the bushel, the dollar, the kilowatt hour, the percent, the person, or an economic index number, etc. That is, the hypernumber unit of that axis or numberline becomes the unit "vector"--the unit _state_ vector--for that state-variable.

We identify units of measurement with "_qualitative_" hypernumber units. This does not imply that a given unit of measurement has the quality of the hypernumber unit assigned to it _intrinsically_, in itself, or in general. A given state-variable does not "have" the quality "i" or "ε" in the _abstract_, independent of the system-context in question. On the contrary, this context determines its hypernumber assignment. A given unit of measurement, such as kilowatt-hour, may be identified in one model with the i-united axis, in another with the ε-axis, in another with the w-axis, and so on. The imputed i-quality, ε-quality, or w-quality of the measurement in these cases is not a _static_ quality "inhering" in it separately, by itself. It does not refer to the particular system quality measured by the quantities of that unit. The hypernumber quality attributed to the measurement in each case refers to its dynamical, _behavioral_ quality, and refers to its _connection_ to the _other_ state-variables _in this dynamical sense_.

Suppose, in a given model, the rate of unemployment is refered to the i-axis, and the rate of inflation to the r-axis. Suppose further that the state of the economy, defined by these two state-variables taken together, is described by a hypernumber reflexive function of the form

$x(t) = g(t)i^{f(t)} \equiv g(t)\cdot[\cos(\frac{\pi}{2}\cdot f(t))\underline{r} + \sin(\frac{\pi}{2}\cdot f(t))\underline{i}]$. The reason the unemployment measure would be referred to an i-axis, the inflation-measure to an r, is dynamical and relative, not static, atomic, and absolute. The reason would be that unemployment varies as a sine-wave-form relative to inflation varying as a cosine-wave-form. The latter varies 90° out of phase with the former, and so on. There would be nothing intrinsically "i-like" about unemployment apart from this dynamical connection to inflation. If the other state-variable were something else, say the interest rate, then we might have to associate the unemployment variable with a w-axis, and treat it as "w-like", for example.

The hypernumber quality attributed to a state-variable is an expression of the coordination of the changes of that state-variable with those of the other state-variables. The hypernumber assignment of each state-variable is derived from the state-space trajectory itself, that is, from the overall pattern of co-variation of the ensemble of state-variables. This pattern is none other than the expression of the dynamical law of the system with respect to the choice of state-variables. That is, this pattern of co-variation, whose pictorial unity is the trajectory, expresses the verb-essence, the behavioral-signature, the law of (evolutionary) motion of the system-process in question.

It is a unique property of the hypernumbers -- that their self-operation generates a continuous pattern of interconnected variation along two or more qualitatively distinct axes--that makes the hyperfunctions work. That is, for example, $x(t) = i_n^t$ maps t into the plane $i_n \perp r$, or ε_n^t maps t into the 4-space $i_n \perp r \perp \varepsilon_n \perp i_0$*.

$$\underline{x}(t) = i_n^t \equiv \begin{bmatrix} a(t)\underline{r} \\ b(t)\underline{i}_n \end{bmatrix} = a(t)\underline{r} + b(t)\underline{i}_n \; ; \qquad \varepsilon_n^t = \begin{bmatrix} a(t)\underline{r} \\ b(t)\underline{\varepsilon}_n \\ c(t)\underline{i}_n \\ d(t)\underline{i}_0 \end{bmatrix} = a(t)\underline{r} + b(t)\underline{\varepsilon}_n + c(t)\underline{i}_n + d(t)\underline{i}_0$$

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*[There are various subvarieties of ε and i, distinguished by subscripts n. See p. 88 below.]

Consider the following trajectory:

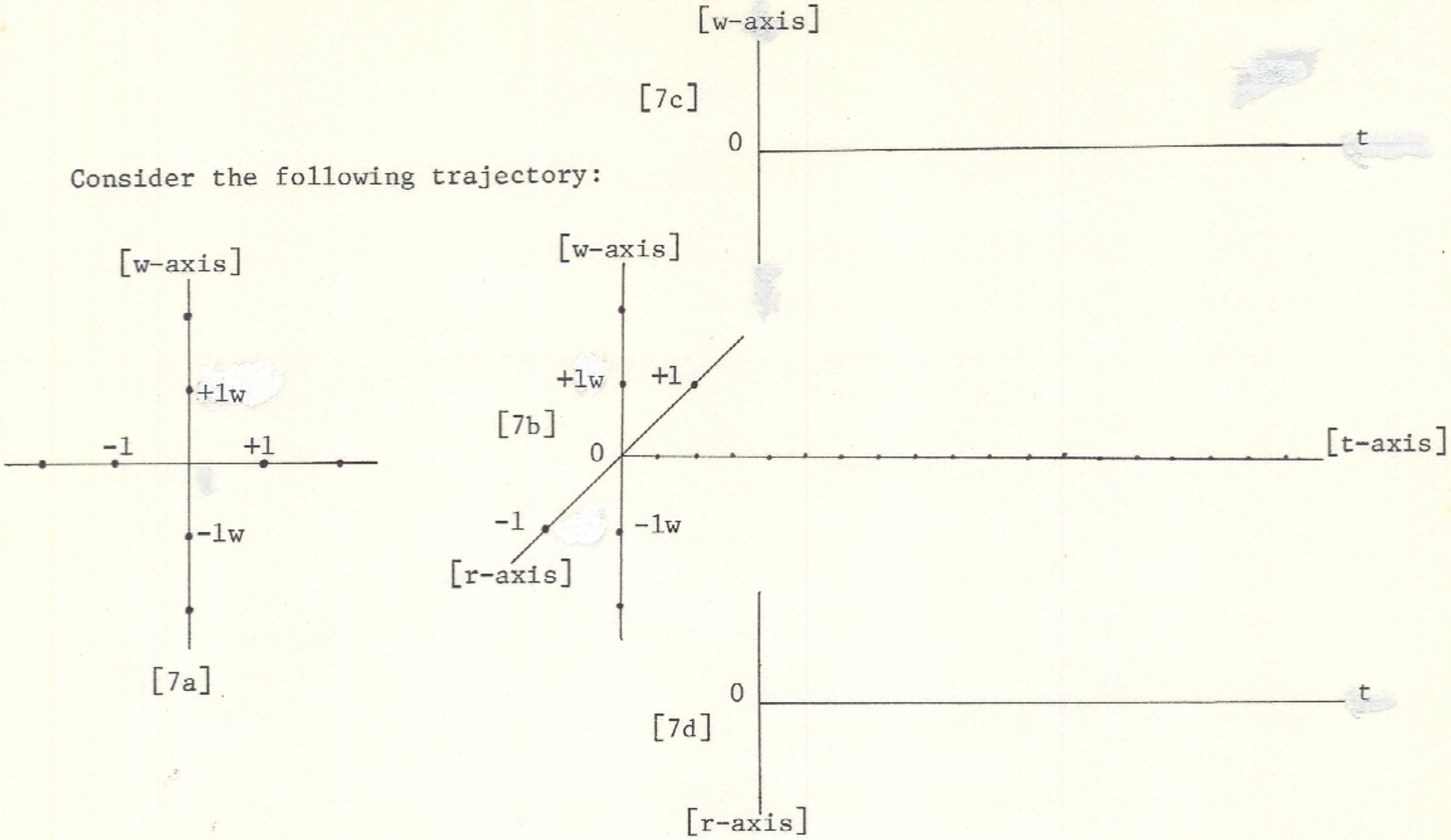

Figure 7. Specimen of a Hyperfunction Trajectory*
(in Timed State-Space)

It is described by the hyperfunction $\underline{x}(t) = \left(\frac{2e^t+1}{2e^t-1}\right) \cdot \left(\cos(\sin\frac{\pi}{3} f(t))\right) \cdot w^t = g(t)w^{f(t)}$ where $g(t) = \left(\frac{2e^t+1}{2e^t-1}\right) \cdot \left(\cos(\sin\frac{\pi}{3} f(t))\right)$ and where $f(t) = t$. Suppose the w-axis in the figure above is assigned to the rate of investment and the r-axis to the rate-of-interest.† Then this state-space trajectory describes a certain economic process. But suppose the w-axis is assigned to blood sugar concentration and the r-axis to the rate of production of thyroxin in a particular individual. Then the trajectory is being used to model a certain physiological process. Or, the hyperfunction might describe the evolution of the state of a star in terms of the rate of star "sunspot" formation referred to the w-axis, and the rate of radiation release from stellar "flares" (flare-ups), referred to the r-axis.

The point is that a single basic trajectory form, such as the one denoted

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*[The power-orbit of (+w) has the formula:

$$(+w)^{\pm n} = (\cos(\pi n/3) \mp (1/\sqrt{3})\sin(\pi n/3))\underline{r} + ((2/\sqrt{3})\sin(\pi n/3))\underline{w} \quad]$$

†[Note the problem of negative values in these examples. This could be redressed by a shift of the origin via addition of suitable constants.]

w, can be used to describe countless distinct processes which exhibit no superficial resemblance to one another. Just so, a single letter of a phonetic alphabet occurs in a great variety of words and sentences which share no obvious semantic commonality.

A letter, a phonetic character, is not immediately a unit of meaning, a semantic unit. Likewise, a hypernumber ideogram like w is not immediately a unit of a definite, recognizeable kind of process. Immediately it refers only to a definite type of trajectory or form of movement in hypernumber space. It is only via the state-space interpretation of that number-space that an operator ideogram like w acquires a definite process-meaning. The assignment of definite measurements or state-variables of a particular system or system-variety to the hypernumber axes framing the space is what finally pins down the process meaning of the characteristic trajectory or power-orbit of a given hypernumber. The hypernumber unity of an axis, once this assignment is made, does not "measure" or express anything "intrinsic" to that system-quality which is being quantified in a unit of measure assigned to that hypernumber axis and unit. Rather, that unity or qualitative unit measures the quality-of-dynamical interconnection of the system-qualities in terms of the state-variables selected. It measures the behavioral relation of the particular state-variable assigned to it, to the other ones chosen.

The different hypernumbers represent different laws of motion, but in a very abstract way. Evolution, organic change or change of organization is depicted as motion via the state-space method. Any process which can be measured --whose changes can be quantified--can be depicted in this way. For a given assignment of axes to measurements, the shape or topology of the state-space trajectory has a clear, process-characterizing meaning. Each hypernumber numeral stands for a distinct trajectory-morphology--rectilinear or fixed-pointal (r), circular (i, ε), elliptical (w), lobate (p,q), cassinian-ovate (m), cornu-spirallic (Ω), etc.

Thus the hypernumbers can be viewed as providing a taxonomy of general systems, classifying systems or processes according to their trajectory type in state-space depiction. Of course, the classification of a given system or eventity depends upon the state-variables selected out of the usually large array of possible ones. Both the number of state-variables selected and their specific qualities affect this classification. Different selections can produce different hypernumber classifications for one and the same system. Still, classification by the hypernumber "system-verb" centering the reflexive function model or solving-function of a system might be the basis of a taxonomy for general system theory. A nonarbitrary classification would require conventions and standards for the selection of state-variables in both number and type in the formal or canonical description of general systems. However, this may not be the point. Words can be classified according to their sound types and sound composition, i.e. according to the phonograms they contain in their phonetic representation. But this does not directly constitute a classification according to meaning. Of course, there are "morphemes" represented typically by characteristic groups of phonograms at a sub-word level. These sound-parts of words do form semantic units. The hypernumber power-orbits represent correlated patterns of vibration. For example, the i-power-orbit represents the correlation of a sine-wave pattern with a cosine-vibration. But the degree to which the hypernumber "system-verb" classification can provide an "ontologically" meaningful taxonomy of general systems is a matter for further exploration.

This trajectory-taxonomy idea, the idea that the hypernumbers form an "alphabet of process", suggests a geometrical formulation of our conjectured "Fundamental Theorem of Nonlinear Algebra". It also suggests a geometrical, topological approach to a proof. To be descriptively complete, a phonetic alphabet must "cover" the associated spoken language. That is, it must have a separate letter or combination of letters capable of describing every significant sound of utterance employed in that language. Analogously, the hypernumber "alphabet" must "cover" all possible trajectory-morphologies which could possibly represent the "track" through state-space of a dynamical system. In particular, the track described by every possible solving-function x(t) of a nonlinear differential equation $D^m x(t) = \ldots$ must be expressible in terms of the "number-movements" induced by unit hypernumber operators, or multiples and additive compounds of these, in mutual interaction ("flexion") and/or in self-action ("reflexion"). In other words, our conjectured "Fundamental Theorem" asserts that flexive and reflexive functions involving the nine hypernumbers plus r are sufficient to formulate every possible meaningful state-space trajectory. Any trajectory pattern should belong naturally to some subspace of hypernumber state-space. For instance, variations on a circular trajectory around the origin belong to the $i \perp r$ plane subspace. Variations on an elliptical theme belong to the $w \perp r$ subspace, and so on. That is, the conjecture asserts that the nine hypernumbers form a set of "choreograms" capable of exhaustively describing the choreographies of the "dances" in state-space of general dynamical systems.

This set of all possible trajectories is a very large transfinite set. But developing the topological arguments demonstrating that the nine hypernumber varieties form a minimal set, necessary and sufficient, for trajectory-description, would be a way of establishing the Fundamental Theorem. It would also strongly corroborate the conjecture that there are no more hypernumbers beyond these nine.

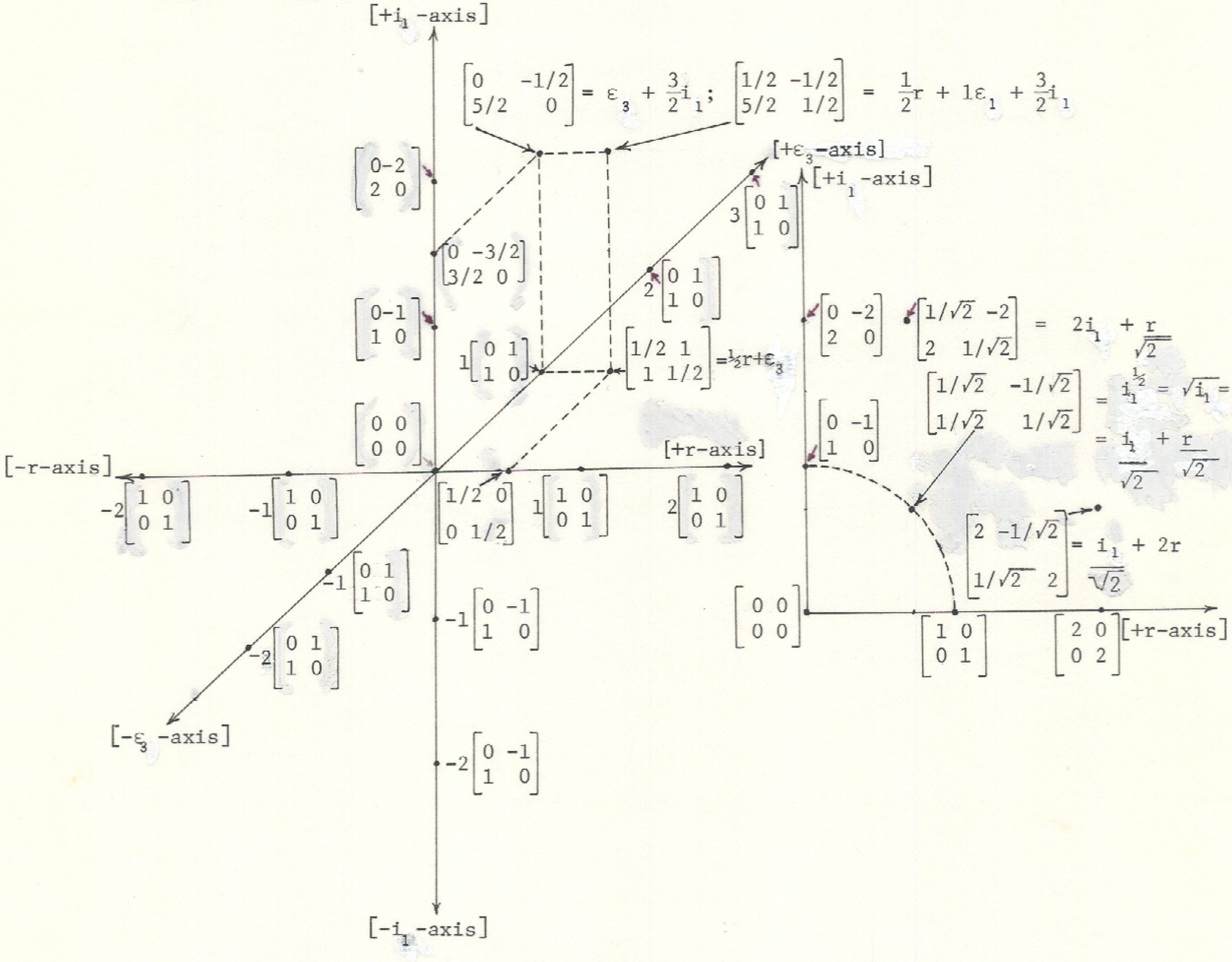
At this point we should note that the "choreographic" facilities of the hypernumbers are not limited to the nine "generic" varieties. Hypernumber state space is not limited to the nine "generic" axes plus the ordinary, r axis. There are subvarieties within generic varieties. In some cases, the number of subvarieties is infinite. The "differentia specifica" are readily visible in the special cases of the hypernumbers which are linear operators, and which can hence be expressed in matrix form. Several varieties of i and ε are so expressed below. They are distinguished by subscripts when given in their non-matrix, "singular" or "unitary" hypernumber scalar form:

$$i_0 = \begin{bmatrix} i_1 & 0 \\ 0 & i_1 \end{bmatrix} ; \quad i_1 = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} ; \quad i_2 = \begin{bmatrix} +i_1 & 0 \\ 0 & -i_1 \end{bmatrix} ; \quad i_3 = \begin{bmatrix} 0 & i_1 \\ i_1 & 0 \end{bmatrix}$$

$$\varepsilon_1 = \begin{bmatrix} 0 & i_1 \\ -i_1 & 0 \end{bmatrix} ; \quad \varepsilon_2 = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} ; \quad \varepsilon_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Every subscript-variety of i--i_n--has the generic properties of i: $i_n^2 = -1$; $i_n^4 = +1$; $i_n^5 = i_n$,etc. Every variety of ε--ε_n--has the generic properties of ε: $\varepsilon_n^2 = +1$; $\varepsilon_n^3 = \varepsilon_n$, etc. Each such sub-variety has its own distinct perpendicular axis or dimension in the full hypernumber space. Thus the fullest (hyper)number-space has many more than the 10 basic dimensions. In fact, it is infinite-dimensional. The full space can accommodate systems which have more than one i-like or ε-like state-variable.

This approach allows us to concretely depict an "abstract space" like the "space of linear operators", or the "space" of matrices. Such function-spaces are conceived in analogy to ordinary "vector spaces" or Real number spaces $\mathbb{R}^n$. In a vector space, every vector corresponds to a definite point. In a function space, then, every function is thought of as associated with a definite "point" also. But such spaces are usually considered undepictable. The actual layout or deployment and "nextness" relations of such function-points are considered unvisualizeable. But, for two dimensional square matrices, the space of matrices or linear functions turns out to be quite renderable. It is just a subspace of hypernumber space:



[$+i_1$-axis]
$\begin{bmatrix}0 & -1/2\\ 5/2 & 0\end{bmatrix} = \varepsilon_3 + \frac{3}{2}i_1$; $\begin{bmatrix}1/2 & -1/2\\ 5/2 & 1/2\end{bmatrix} = \frac{1}{2}r + 1\varepsilon_1 + \frac{3}{2}i_1$
[$+\varepsilon_3$-axis]
[$+i_1$-axis]
$\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$
$3\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
$\begin{bmatrix}0 & -3/2\\ 3/2 & 0\end{bmatrix}$
$2\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$
$1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
$\begin{bmatrix}1/2 & 1\\ 1 & 1/2\end{bmatrix} = \frac{1}{2}r + \varepsilon_3$
$\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$
$\begin{bmatrix}1/\sqrt{2} & -2\\ 2 & 1/\sqrt{2}\end{bmatrix} = 2i_1 + \frac{r}{\sqrt{2}}$
$\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix} = i_1^{\frac{1}{2}} = \sqrt{i_1} = \frac{i_1}{\sqrt{2}} + \frac{r}{\sqrt{2}}$
$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$
$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$
[-r-axis]
[+r-axis]
$-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$
$-1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$
$\begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}$
$1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$
$2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$
$\begin{bmatrix}2 & -1/\sqrt{2}\\ 1/\sqrt{2} & 2\end{bmatrix} = \frac{i_1}{\sqrt{2}} + 2r$
$-1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
$-1\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$
$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$
$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$
$\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$
[+r-axis]
$-2\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
[$-\varepsilon_3$-axis]
$-2\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$
[$-i_1$-axis]

[Figure 8: 2 by 2 Matrix Operator Space as an Excerpt From Hypernumber or Generalized Number Space]

The Space of Processes

The higher hypernumbers, beyond i and ε, are not linear operators and hence cannot be expressed as matrices. But this interpretation of the space of two-by-two linear functions $f \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ reveals something about hypernumber space as a whole. We will designate that space $\mathbb{H}$, to distinguish it from $\mathbb{R}^9$ or $\mathbb{R}^n$ [the space of n mutually-perpendicularized r axes: $\underbrace{\mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \times \ldots}_{n \text{ times}} \times \mathbb{R}^1$[*] $\equiv$ $(\ldots(((r_1 \perp r_2) \perp r_3) \perp r_4) \perp r_5 \ldots \perp r_9) \ldots \perp r_n)$].

$\mathbb{H}$ is a function-space. Each of the hypernumber unities i, ε, w, p and so on is a function like the f in $y = f(x)$. The points in the "point-set" or "space" $\mathbb{H}$ are functions; are named by these functions. For example, the point i represents the function of 90° rotation in $i \perp r$ space. The point r represents the identity function. That is, $\mathbb{H}$ is a space of operators. Every point in $\mathbb{H}$ stands for a definite operation.

$\mathbb{H}$ is also a space of numbers. Every point in the set $\mathbb{H}$ is a definite (hyper)number value. This formulation and the preceding one are identical. $\mathbb{H}$ reveals a conceptual fact that has been obscured by exclusive experience with the vector-space $\mathbb{R}^n$. A number is a function. A function is a number. An operation is also a value. A value is also an operation. It makes sense to write $h = g(f)$ where h, g, and f are all functions. That is, in $y = f(x)$, y, f, and x can all be regarded both as functions and as (generalized) numbers. Hypernumber "multiplication" or flexion is a form of "functional composition": $iw \equiv i \cdot w \equiv i \circ w \equiv i \times w$. In the form $y = f(x)$, f is to be regarded as a coefficient of x, and y as the product of $f \cdot x$.

A point in $\mathbb{H}$ is labeled or "named" by any operation, any interaction, any number-process that produces it. Thus, along the r axis, the point most briefly designated "2" can also be denoted $\frac{4}{2} = 2^{-1}(4)$, $(1 + 1)$, $(\sqrt{2})^2$, etc. Any point thus has many names, for there are many different ways of "getting to" that point from other points in $\mathbb{H}$. Thus the "=" sign does not signify equality of operations or movements, but merely cotermination: identical end points.[61] In $2^{-1}(4)$, $+\sqrt{4}$ and $\sqrt{2}(\sqrt{2})$ we have three very different operations and movements. But $+\sqrt{4} = 2^{-1}(4) = \sqrt{2}(\sqrt{2})$ because all three end up at the same place in $\mathbb{H}$.

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[*] [Self-"multiplication" of the sets $\mathbb{R}^1$ here means the "Cartesian product" of the sets, which forms sets of ordered "n-tuples" from the elements of the sets so "multiplied".]

In terms of "number-movements", $ab \rightarrow c$, a point in $\mathbb{H}$ can be named "ab" or "c"--have "ab" as one of its "synonyms"--for every movement a(b) that *ends up* at it. The end-point of a movement takes over the name of the whole movement. Thus every point in $\mathbb{H}$ represents a "process"--actually many processes--in the sense of our "process-algebra". $\mathbb{H}$ is the "space of processes", "process-space" or "operation space"--a space of "verbs". Interpreted as a state-space, its points represent states not merely statically, as "values", but as states *of operation* (states of activity). The *values* expressing the states of systems in $\mathbb{H}$ are also *operators*, operations. The state-"vector" valued function $\underline{x}(t)$ is, in $\mathbb{H}$, an *operator*-valued function; indeed, a *function*-valued function!

The general process-expression $ab \rightarrow c$ or, equivalently, $fx \rightarrow y$ or $y = fx$, is constituted as follows. The point x or b is the starting-point of the movement. As just discussed, y or c is its terminus. The symbol $\underline{a}$ or f stands for the "movor" or "motor" of the process; the operation which, applied to x or b, takes you to y or c. $\underline{a}$ or f, that is, stands for the general shape of the movement; for the "trajectory-type" which is to be applied to the transit between b and c; for the movement *connecting* b and c; the one which, applied at b, ends up at c. In the example of *Fig. 7*, x is the point named +2i, y is -2r, and f is i, which stands for a circular-shaped movement through 90° of arc in the $i \perp r$ plane.

But wait a minute! The symbol i also denotes a *point* in $\mathbb{H}$. How can a single symbol designate both a (*type of*) *movement* and a *single point*?

The case of *reflexive* process--of reflexive functions--brings this problem to sharpest clarity. The reflexive case, recall, is that where, in $ab \rightarrow c$, $a = b$; in $y = fx$, where $x = f$: $f(f) \rightarrow y$; $y = f^2$. In temporalized form, this becomes $a^t b \rightarrow c$ where $a = b$: $a^t(a) = a^{t+1}$, or $y = f^t = f^{t-1}(f)$. [Since t is a variable anyway, we might just as well rewrite these $a^t \rightarrow c(t)$ and $y(t) = f^t$.] Consider the "reflexion" $i(i) \rightarrow -r$. According to our interpretation above, the two occurrences of i stand for different things. The first i, the rightmost i, the i in the parentheses, the "argument" or "operand", stands for the point named i or +1i. The second i, the leftmost i--i as "function" or "operator"--stands for a movement of 90° circular rotation.

This apparent ambiguity might not seem so bad if all we ever had to consider were interacting pairs of operators. But we must also consider "strings":

$$pw\varepsilon i \equiv p(w(\varepsilon(i)))$$

which represent complex movements inside $\mathbb{H}$. Here we are constantly shifting our point-of-view, from "movement" to "point" back to "movement" again, in "relay" fashion as we carry out the hyperarithmetic computation indicated. The rightmost symbol, i, stands for a point with respect to or from the point of view of the symbol next to i on its left, ε. Likewise, with respect to i, that symbol ε stands for a movement to be applied starting at point i: $\varepsilon(i)$. This interaction of ε and i results in a point again. We then take the endpoint of the process $\varepsilon(i)$ and apply the movement w to that point, (εi), ending at another point $(w(\varepsilon i))$ to which we apply the movement p, and so on.

The resolution of this ambiguity reveals the secret of the process arithmetic. It clarifies the paradoxical necessity of the r-numbers in $\mathbb{H}$. The space of operations must include the axis of the not-operator, the non-movement, r, as one of its dimensions; must include nonoperation as one of its operations; must contain the Real numbers to make the hypernumbers work.

The resolution of this ambiguity is as follows.

The *rightmost* i in $i(i) \rightarrow -r$ can also be grasped as denoting a movement, not just denoting a point. It is the movement $1(i) \rightarrow i$ or $ri \rightarrow i$ that begins at i *and* also ends at i. Or, it is the process $i(r) = i(1)$, also ending at the point i. Thus $i(i)$ can be rewritten as

$$i(i(r)) \quad \text{or} \quad i(r(i))$$

The existence of the identity operator and the "identity point" r in $\mathbb{H}$ enables us to have a movement with the same name as its endpoint: $i(r) \rightarrow i$; or to have a "movement" which stays in place and thus ends where it began: $r(i) \rightarrow i$. Thus i can be both an operator "*on*" the space $\mathbb{H}$ and a point "*in*" it. Thus i can operate upon itself.* The point in $\mathbb{H}$ named after a movement in $\mathbb{H}$ is the point you get to when you apply that movement to r, that is, starting at the point +1. By operating upon (or "starting at") the point named after itself, a movement can operate upon itself; move itself according to itself. Reflexive functions are possible because of the "irreflexivity" of 1; because its power-orbit is a single point; because its movement is no movement, because r has no effect upon itself or upon other operators. Its specific effect *is* to have no effect.

This unity of number and operation, of value and function, of function and argument, of point and movement, is the key to the implementation of the "process algebra" idea. It is the generalization of the "operator" concept already emergent within mathematics, long before the discovery of the hypernumbers beyond i. It is the ideographic expression of the "eventity" ontology, and of the grammatical principle of noun-verb fusion.

It means that, in the general process-form $ab \rightarrow c$, ab represents an event or process, and a and b by themselves already represented processes. It means that $ab \rightarrow c$ is a process made up of processes. It means the interaction "ab" is a *process-valued process of processes*; an operation-valued operation of operations; a function-valued function of functions. That is, the space $\mathbb{H}$ is closed within

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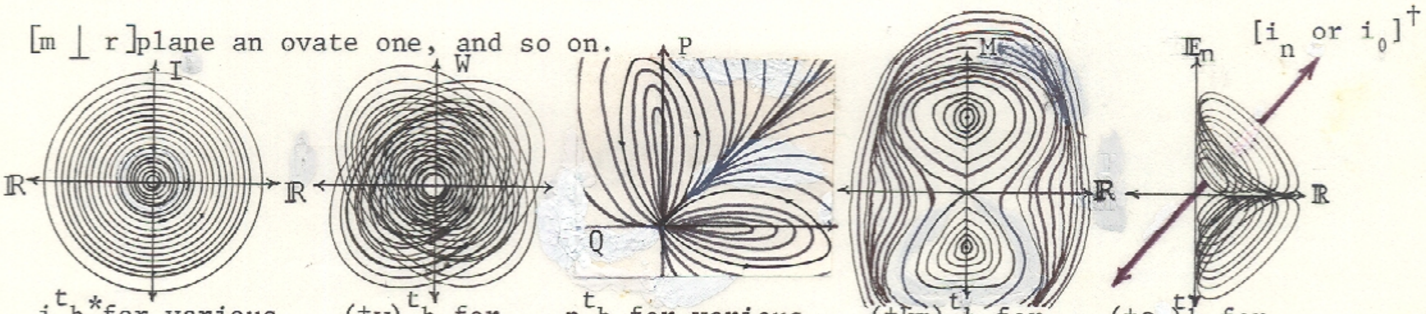
*[This overcomes the mathematical aspect of Wittgenstein's objection to the concept of reflexive functions in *Tractatus Logico-Philosophicus*.[62] See Section 3.333; also 5.25 to 5.2521.]

itself, self-contained. Any composition of functions in $\mathbb{H}$ results in another function in $\mathbb{H}$. The result of the interaction of operators in $\mathbb{H}$ is another operator in $\mathbb{H}$. All movements that begin in $\mathbb{H}$ end in $\mathbb{H}$, and stay in $\mathbb{H}$ inbetween. No process made from "ingredients" in $\mathbb{H}$ leads out of $\mathbb{H}$. No process leaves the space of process. For processes, as defined, $\mathbb{H}$ has no outside.

Such operational closure is a necessary condition for our conjectured "Fundamental Theorem of Nonlinear Algebra" to be true. It is necessary, in particular, if general nonlinear differential equations are to find their solutions exclusively within $\mathbb{H}$, interpreted as a state-space.

What, then, are the hypernumbers? We suspect as follows. They are flows. They are a codification of the basic types of flow-pattern possible in space. They form a space, $\mathbb{H}$, which is an ORGANISM of different planes (and other subspaces), each representing one of the basic possible flow-types. For example, the $[i \perp r]$ plane within $\mathbb{H}$ has a circular flow-field. The $[w \perp r]$ plane has an elliptical flow-structure. The $[p \perp q]$ plane has a lobate "grain", and the $[m \perp r]$ plane an ovate one, and so on.



$i^t b$* for various b in the $[i \perp r]$ subspace of $\mathbb{H}$

$(\pm w)^t b$ for various b in $[w \perp r]$

$p^t b$ for various b in $[p \perp q]$

$(\pm km)^t b$ for various b in $[m \perp r]$ and for various k

$(\pm \varepsilon_n)^t b$ for various b in $[r \perp \varepsilon_n \perp i_n \perp i_0]$ space within $\mathbb{H}$

[Figure 9: Operational "FIELDS" or "FLOWS" of some Hypernumber Operators]

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*[When t is allowed to vary beyond t = 1.0, a function of the form $f(t) = u^t b$ is somewhat like a continuous version of a Markov process.]

†[We have here actually a 4 dimensional space, and either i_n or i_0 could be the third axis as shown.]

The 4-dimensional $\varepsilon_n \perp i_n \perp r \perp i_0$ subspace has a skewed-circular "grain". Each hypernumber operator u, in a temporalized process-expression of general form $u^t b$, is like a "tugboat". It "pushes" the state-point, originally resting "on" the point named "b", around in the space in a characteristic pattern intrinsic to the particular hypernumber u in question. The hypernumbers thus may provide a new, more explicit, and more general rendering of the concept of the "flow" of a differential equation, including the flows of *nonlinear* differential equations.

The Arithmetic of States

When ℍ, or any subspace thereof, is interpreted as a state-space, every arithmetic process represents a *dynamical event*, a process of change of state. Every arithmetic expression, once evaluated, has as value a definite state, the stopping-point or resting-place of a *trajectory within and through* ℍ, a state-space *trajectory*. For every hypernumber value represents--*unambiguously*--a definite *point* in ℍ. Every *point* in ℍ, or in any subset of ℍ, once that point-set or subset has been interpreted as a state-space, represents a definite *state*. It represents a definite state of that system to which the state-space in question pertains.

Generalized arithmetic, hyperarithmetic, becomes *a calculus of states*. More dynamically stated, it becomes a calculus of state-*trajectories* or of *state-transitions*. Since we use trajectories to represent processes, we have indeed a *calculus of processes*.

This is accomplished via (i) hypernumber analytic geometry, that is, via the geometric interpretation of hyperarithmetic; by the concept denoted ℍ, and; (ii) the state-space interpretation of this geometric interpretation of arithmetic: the concept of *hypernumber state-space*. By (i), every calculation, every computation is depicted as a *movement* in (generalized) number-space. An arithmetic process or computation is the transformation of one *value*(s), the "input", into another value(s), the "output" or "answer". Geometrically, this process of calculation is represented by a movement from one point in number-space to another point in that space. By (ii) this "jump" from point-value to point-value is utilized to represent a jump from one state of being to another state of being for that system whose development is being portrayed using the space.

Thus the formulas and expressions of generalized arithmetic come to represent state-processes, i.e. trajectories. Computation comes to describe system dynamics. Arithmetic becomes a calculus of system transformation and metamorphosis, a calculus of evolution.

The states described by points in hypernumber state-space are not mere inert, passive values, mere "nouns", as they appear to be when $\mathbb{R}^n$ is used as the state-space.

A point in $\mathbb{R}^n$, a state in "$\mathbb{R}$eal" state-space, is a "vector". This vector is represented by a list or array containing the n coordinates of the point "pointed out" by the arrow from the origin which depicts the vector. But this list is only the name of the point. A vector is not an operator "on" $\mathbb{R}^n$. If $n \neq 3$, the vector product of two vectors is not defined.[63] The scalar or "inner" product of two vectors won't get you around in $\mathbb{R}^n$--it will only throw you back into $\mathbb{R}^1$, that is, at best, onto a designated one of the n $\mathbb{R}$eal number-lines framing the space. At worst, the scalar product throws you out of the space altogether, to a separate space designated $\mathbb{R}^1$. That is, $\mathbb{R}^n$ vectors or points ("pointors") are not also "verbs". They are not also operators "on" the space $\mathbb{R}^n$ in addition to being "nouns" or names for $\mathbb{R}^n$ points. They won't move you around in $\mathbb{R}^n$. Their flexions and reflexions will not produce/describe trajectories or movements through $\mathbb{R}^n$. The entities which do induce movements through $\mathbb{R}^n$ are the n x n matrices. These matrices come from outside $\mathbb{R}^n$. They are interpreted geometrically as points residing in a matrix or linear operator space which exists "elsewhere"; in disconnection from $\mathbb{R}^n$.

The states expressed by hyperarithmetic expressions are also operators in the state-space where they reside. Hyperarithmetic expressions are operation-valued expressions. A hypernumber state-value indicates more than just a position in the hypernumber state-space. It indicates the kind of continuum, the kind of trajectory to which it belongs, in the context of the flexive and/or reflexive structure of the arithmetic expression from which it arises.

A hypernumber state-value, a point in $\mathbb{H}$, thus represents a state of operation, a state of activity of the system modeled. Hyperarithmetic expressions give the states of systems in the form of "verbs". The state of a system at a given moment is the "verb" (operator) which describes its activity at that moment. $\mathbb{H}$ is not just a space of points or positions. It is a space of verbs. That is, $\mathbb{H}$ is a space of operations, a space of "processes".

We have defined "process" as "state-process". A process in the world is modeled by a track or trajectory in state-space; a movement-pattern or "dance" in number-space, where this number-space has been interpreted as the point-set or space "of all possible states" of a definite system or "eventity". Thus hypernumbers, interpreted as states, represent "states of process" or "process-states". The arithmetic of hypernumbers, so interpreted, is a "process-arithmetic". The algebra of this arithmetic is a "process-algebra". Because σ is a hypernumber, this algebra--"hyperalgebra"--contains the entirety of "analysis", the entire calculus of integrodifferential equations.[64] Hyperalgebra is also Nonlinear algebra, containing all of linear algebra as a special case. If our conjecture is correct, Hyperalgebra provides a calculus of states and processes which is complete. Thus the hypernumbers implement the "process algebra" we sought at the outset of this section.

Specimens of the Process-Arithmetic

A few examples will illustrate the summary propositions foregoing. Even a single hypernumber standing alone

$$w$$

denotes, unambiguously, a definite point in $\mathbb{H}$. With $\mathbb{H}$ interpreted as a state-space for some system, this point stands for a definite state of that system. Thus arithmetic becomes a way of talking about (definite) points in many-dimensional space; a way of talking about geometry; a way of talking about graphs, trajectories, and figures ("point-sets"). Via the state-space interpretation of this geometry, it becomes a way of talking about states, process-states, states of operation. Via temporalization, or t-exponentiation, the process arithmetic becomes a way of talking about state-trajectories, about whole patterns of changes of state ; that is, about processes. This is by virtue of the fact that the assertion of the single symbol "w", for example, designates the unique point which, in full detail, is given by

$$0r + 0i + 0\varepsilon + 1w + 0p + 0q + 0m + 0\Omega + 0v$$

which indicates the point on the w-axis one unit up; the point whose coordinates with respect to all other hypernumber axes are zeros. Notice this does not work for $\mathbb{R}^n$. To write simply

$$1$$

does not specify unambiguously a point in $\mathbb{R}^n$, say the point one unit up from the zero point on the first, x-axis. It could mean the "one" point on any of the n Real axes framing the space. It would be necessary to introduce vector-array notation:

$$\begin{bmatrix}1\\0\\0\\\cdot\\\cdot\\\cdot\\0\end{bmatrix} \neq \begin{bmatrix}0\\1\\0\\\cdot\\\cdot\\\cdot\\0\end{bmatrix} \neq \begin{bmatrix}0\\0\\1\\\cdot\\\cdot\\\cdot\\0\end{bmatrix}$$

or unit-vector notation

$$1\hat{x} \neq 1\hat{y} \neq 1\hat{z}$$

$$1\hat{x} \equiv 1\hat{x} + 0\hat{y} + 0\hat{z} + \ldots$$

to remove the ambiguity, that is, to distinguish the various "one" points on the various axes one from another. For, operationally, they are all just "r". The isotropic character of $\mathbb{R}^n$ causes the difficulty. (It can be argued that unit-vector notation, and the vector "multiplication" operations, are historical and conceptual vestiges of Hamilton's "quaternions". The quaternions, or 4-complexes $a + b\underline{i} + a\underline{j} + d\underline{k}$, in turn, form just a subspace of $\mathbb{H}$, namely the subspace $r \perp i_1 \perp i_2 \perp i_3$. That is, $a + b\underline{i} + c\underline{j} + d\underline{k} = ar + bi_1 + ci_2 + di_3$, or $\underline{i} = i_1$; $\underline{j} = i_2$, and $\underline{k} = i_3$).

We see then that the anisotropic character of $\mathbb{H}$ is what gives its arithmetic the power to speak unambiguously about points and states. The different axes represent qualitatively different operations. This operational anisotropy and variety is also what gives the $\mathbb{H}$ arithmetic its rich repertoire of trajectory-describing powers, lacking when $\mathbb{R}^n$ is presumed to be the exclusive habitat of state-processes.

More elaborate expressions, expressing hyperarithmetic computations, likewise designate simply single points or states. For example, recalling that, $(\forall n)\ i_n\varepsilon_n = i_0 = \varepsilon_n i_n$, the expression

$$(4i_1)\cdot(3i_1 + \frac{1}{2}r + \sqrt{2}\,\varepsilon_1) = -12r + 2i_1 + \sqrt{32}\,i_0 + 0\varepsilon_1$$

designates the point twelve units down the r axis, two units up the i_1 axis, $\sqrt{32}$ units up the i_0 axis, and zero units up (or down) the ε_1 axis.

This expression contains additional information. It can be viewed as specifying a movement between two points, $P_1 \equiv (3i_1 + \frac{1}{2}r + \sqrt{2}\,\varepsilon_1)$ and $P_2 \equiv (-12r + 2i_1 + \sqrt{32}i_0)$. It specifies not only the end-point P_2, but the starting point P_1, plus a path connecting the two, a definite *way of getting to* P_2 (*from* P_1). This path-shape is symbolized by the operator $4i_1$.

The end-point or final value specified in the following expression is the same as that of the above

$$(\sqrt{32}\,r)\cdot\left(\frac{-12r}{\sqrt{32}} + \frac{2i_2}{\sqrt{32}} + i_0\right) = -12r + 2i_1 + \sqrt{32}\,i_0$$

But a very different movement is described. It begins at

$P = \frac{-12}{\sqrt{32}}r + \frac{2i_2}{\sqrt{32}} + i_0$, not at P_1. The path is a rectilinear "stretching" from P to P_2. For $\sqrt{32}\,r$ is a rectilinear operator--stands for a rectilinear movement--whereas $4i$ symbolizes a curved path, a rotation. Thus the final value of an expression is not the whole story.

Suppose $r \perp i_0 \perp i_1 \perp \varepsilon_1$ is being used as a state-space. Then each axis has been designated the scale of a definite measurement on the system being modeled. Suppose it is the economic system, and the r-axis is to map the inflation-rate, the i_1 axis the rate of growth-rate of GNP, the i_0 the rate of unemployment, and the ε_1 the rate of investment. Then the first expression represents a transition in the state of the economy from a situation with a $\frac{1}{2}$% annual inflation rate, a 3% GNP growth rate, about a 1.414...% rate of capital investment, and zero unemployment, to a situation with 12% *de*flation, 2% growth, about 5.65% unemployment, and zero investment-rate. The second expression represents a jump *to* the same state, but *from* a very different previous operating point, with approximately 2.12% deflation, nearly .35% growth, 1% unemployment and zero investment-rate.

Application of the Process Algebra: The "Semantics" of $\mathbb{H}$

Once a given assignment of state-variables to the axes of $\mathbb{H}$ has been posited, $\mathbb{H}$ has been interpreted for the purpose of describing a dynamical system. $\mathbb{H}$ is being used as a state-space. It is being used to describe the life-history of a certain *type* of system. The space so interpreted, the system so described, determine a "universe of discourse", as it were. They determine the *semantics* of all hyperarithmetic expressions ensuing under their rubric. Every conceivable arithmetic expression -- at least, every one involving (hyper)-numbers belonging to interpreted axes -- suddenly acquires *meaning* with respect to this dynamical system. In symbolic logic, a formal proof appears as a sequence of formulae, each constrained to be a *true* formula, deducible from the formula immediately preceding it, or from others assumed true or proved previously. It is similar for sequences of computative expressions ensuing under a given dynamical system interpretation of $\mathbb{H}$. Each such expression becomes a "state-ment" about the system in question. It asserts a definite state (symbolized by a unique point in $\mathbb{H}$) "about" the system. Or, such an expression describes a definite state-*process* of that system. It describes a transit from one state (point) to another, that is, a portion of the system's state-space trajectory.

Notice that no explicit temporal information is provided by these forms. At this level of application, the process-arithmetic speaks of states and state-transitions in a "*metatemporal generality*" -- that is, in *abstraction* from any particular chronometry or time-relationship. The exact timing is left open. Only the sequence or direction of the changes in state--the chronology--is explicit. But these expressions can be temporally particularized, by superscripting monotonically increasing time-functions in the ways we have described.
The "pure arithmetic" expressions, without temporal specificity, abstract from temporality in the same way formal logic does. Indeed, such expressions can be thought of as formulating a "logic of events"; a logic of action, or "logic of dynamics." When the time dimension of the expressions is made explicit, this logic becomes *temporalized*.

The Fate of the "Four Basic Operations" of Ordinary Arithmetic in the Generalization to the Process-Arithmetic:

The Syntax of an Operatorial Ideography

Ordinary arithmetic is traditionally regarded as involving four basic operations -- addition, subtraction, multiplication, and division. Our principle of juxtaposition seems to pertain to and to generalize merely one of these, multiplication. What of the other three? Do they still obtain and remain as separate, "pure" operations as before? Does the principle of noun-verb, quantity-operation fusion leave them untouched? Or do they too join in the conceptual unification when a calculus whose every symbol is an operator -- an "operational ideography" -- is implemented?

In answering these queries, the main conceptual stumbling block is encountered with the case of addition. After that case is resolved, the other two fall rapidly into place.

The plus sign is still employed in the process-algebra, as the specimens foregoing attest. But, rather than regarding it as an operation, we grasp it as an *explicit* symbol of *non-operation*. That is, the process-algebra employs '+' as a *separator*. The '+' symbol serves as the explicit indicator of *non*-juxtaposition. It is used to indicate the formation of a "collective operator".

Let us explicate the foregoing remarks by a syntactic exercise. When two hypernumeric operator symbols, say ε_1 and i_1, occur "on the page" under the scope of a given metrical assignment of the axes of $\mathbb{H}$, that is, under a given state-space "interpretation" of $\mathbb{H}$ (thus both refering to the same, specific dynamical system) two relationships between them are possible. Either they are juxtaposed, indicating interaction: $\varepsilon_1 i_1$ or $i_1 \varepsilon_1$. Or, they are "summed":

$i_1 + \varepsilon_1 = \varepsilon_1 + i_1$. They are explicitly decoupled and separated by the '+' symbol; overtly non-juxtaposed. In either case, an unambiguous assertion is made regarding the state of the system in question. In the first case, the state* $\varepsilon_1 i_1 = i_1 \varepsilon_1 = i_1$ is attributed to the system at the moment of time in question. That is, writing

$$\varepsilon_1 i_1$$

or

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declares that the state-space trajectory passes through i_0, the point one unit out from the origin on the positive side of the i_0 axis in $\mathbb{H}$, at the moment in time under discussion. Writing

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declares that the system-trajectory passes through the point whose "coordinates" are $+1\varepsilon_1$ and $+1i_1$ (with all other coordinates implied to be zeroes). That is, asserting "$\varepsilon_1 + i_1$" about the system indicates that the point one unit out from the origin along the positive branch of the ε_1-axis, and one unit out along the positive branch of the i_1-axis, and in the $i_1 \perp \varepsilon_1$ plane, represents the state of the system for the moment of time under discussion, given the metrical, state-variable assignment of these axes under the state-space intepretation of $\mathbb{H}$ then in force.

To simply assert these two symbols on the page, not juxtaposed but separated only by an indeterminate gap of blank spaces:

$$\varepsilon_1 \qquad i_1$$

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*[In 2 by 2 matrix form: $\varepsilon_1(i_1) = i_1(\varepsilon_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \bullet \begin{bmatrix} 0 & +i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & +i \\ -i & 0 \end{bmatrix} \bullet \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = i_0$.]

would be an ambiguous or schizophrenic attribution of state to the system. Such a declaration would leave the connection between the two symbols indeterminate. Or, it would assert that one and the same system had *two* different states at one and the same time: the state represented by the point ε_1 =

$$0r_1+\ldots+0i_1\ldots+1\varepsilon_1+\ldots 0w + 0p + 0q + 0m + 0v$$

and the state represented by the point i_1 =

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The latter would imply two separate trajectories in the same state space for the same single system. At best, this could be used to indicate "bifurcation" of solutions which does give rise to several *possible* trajectories (though, in any particular case, the system "chooses" and follows only *one* of the possible branches). Even then, further labelling would be required to make sense of the assertion. Barring that, the ambiguity can be resolved in one of only two ways. The indeterminate connection between the two symbols must be determined to be either '+' or '•' (juxtaposition), "addition" or "multiplication".

The apparent mystery of the"not-operation" interpretation of the addition "operation" arises from biases inculcated by our restricted experience with $\mathbb{R}$ -only arithmetic. We are as used to the form

$$a + b \rightarrow c$$

as to the form

$$ab \rightarrow c \qquad \text{or} \qquad a \cdot b \rightarrow c.$$

Therefore it seems, first off, that '•' and '+' are just two *different operations* by which numbers *a*, *b* can interact to produce a third number *c*.

But that is only because, with all quantities tagged with the r quality, i.e. with all points confined to the $\mathbb{R}$ axis -- for example:

$$2 + 3 = 5$$

$$\text{i.e. } 2r + 3r = 5r$$

--units of a (2) and b (3) are of like kind, and amalgamate or "stack up" along that axis, hence can be rewritten as a singular arithmetic symbol "5" (or 5r). However, in the general case in hyperarithmetic, the form

$$a + b \rightarrow a + b$$

$$\text{or } \quad a + b = b + a$$

is all that can be asserted. Hyperarithmetic addition is, in general, "nonamalgamative".[65] It is "nonamalgamative" because the "sums" involved are, in general, inhomogeneous. That is, the sums mix qualitatively different units "of measure", qualitatively different hypernumber unities. Sums like

$$5i_1 + 3\varepsilon_1$$

$$3w + 4p + 2r$$

$$i_1 + i_0$$

$$ar + bi$$

cannot be reduced further. The separate terms cannot be "amalgamated" or united into a single elemental[66] arithmetic term.* The units do not agree.

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*[Of course, such a "sum" can be replaced by a singular term in the shape of an algebraic variable, as the convention $z = a + bi$, familiar from the theory of functions of a complex variable, attests. But this algebraic amalgamation does not over-ride the arithmetic inhomogeneity, as manifested, for example, in the restrictions regarding "analytic functions" in the Complex domain.]

Five units of i_1 and 3 units of ε_1 do not add to 8 units of i_1, nor 8 units of ε_1, nor 8 of $\varepsilon_1 i_1 = i_0$. Despite colloquial quips concerning "adding apples and oranges", inhomogeneous sums can be quite useful and meaningful in the mathematical sciences.[67] The manifold applications of Complex numbers and the theory of functions of a Complex variable is perhaps the most dramatic demonstration of this to date.

Of course, the special cases within hyperarithmetic wherein all quantities (multiplicities) involved are tagged with identical units (hypernumber unities) -- homogeneous sums of hypernumbers -- do exhibit amalgamation:

$$5\varepsilon_1 + 3\varepsilon_1 = 8\varepsilon_1$$

$$30w + (-10w) + 4w = 24w$$

$$6r + (-3r) = 3r$$

$$5i_1 + 10r - 3i_1 + 6r = 16r + 2i_1 .$$

Note, however, that every such hyperarithmetic expression, homogeneous or inhomogeneous, amalgamative or not, refers to or "picks out" a definite, unique point in $\mathbb{H}$, and therefore a definite, unique system state, whenever $\mathbb{H}$ has been interpreted as a state space.

Besides Complex arithmetic,* the deployment of unit vectors in Real vector arithmetic provides our most familiar precedent for calculative procedures similar to those exemplified above. In $\mathbb{R}^3$, for example, we often denote the unit vector of the first Real axis by the symbol $\hat{x}$ ($\equiv r_1$ in the hypernumber notation), that of the second by $\hat{y}$ ($\equiv r_2$), and that of the third by $\hat{z}$ ($\equiv r_3$).†

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*(which is actually a subcase of hyperarithmetic, belonging to a subspace of $\mathbb{H}$.)

†(See p. 99 above.)

We then represent points* in $\mathbb{R}^3$ by nonamalgamative, inhomogeneous sums like the following:

$$3\hat{x} + 2\hat{y} + 5\hat{z}.$$

However, in the special case of multiplicities qualified by the same unit vector "tag" -- that is, in the case of homogeneous sums -- amalgamation does occur:

$$3\hat{x} + 6\hat{x} + (-2\hat{x}) = 7\hat{x}$$

$$6\hat{x} + 9\hat{y} + 3\hat{x} + \hat{x} + 2\hat{z} = 10\hat{x} + 9\hat{y} + 2\hat{z}.$$

Overall, the best way to conceive "addition" in its generalized, hyperarithmetic meaning, is as the formation of a 'collective operator', and as the indication of noninteraction among the individual operators which are the internal components of that collective operator. By a collective operator, we mean an array or sum of qualitatively distinct operators which are treated as a composite whole. That is, they operate as a single entity, a single operator, upon other operators. The collective operator acts as a united whole externally, but its component operators do not interact among themselves internally. Complex arithmetic provides the most accessible examples of 'collective operators'. For instance, the Complex number 5 + 3i ($5r_1 + 3i_1$ in hypernumber notation) is a collective operator, representing a definite point in the Complex plane. When 5 + 3i interacts with another Complex number, say 2 + 4i, it is applied to that number as a composite whole:

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*(that is, vectors -- often depicted geometrically as "arrows" pointing to or pointing out said points.)

$$\begin{aligned}(5 + 3i)(2 + 4i) &= 5(2) + 5(4i) + 3i(2) + 3i(4i)\\ &= 10 + 20i + 6i + 12i^2\\ &= (10 - 12) + (20 + 6)i = -2 + 26i \ (= -2r_1 + 26i_1).\end{aligned}$$

But inside the parenthesis which encloses 5 and 3i, collecting them together into a single, "compound" operator, there is no interaction. The 5 does not operate upon the 3i, nor vice versa. Nor does the 5 alone, or the 3i alone, operate when an external operator such as (2 + 4i) is juxtaposed. Both the 5 and the 3i act upon (2 + 4i). And they act upon the whole of it, not just upon 2 or 4i. The 5 and the 3i act together, in unison, as if a single entity. The "+" sign binds them together. That is, it signifies they are to be treated as one entity for purposes of calculation.

In the process arithmetic, a collective operator is a "compound" hypernumber. A compound hypernumber is made up of "elemental"[68] hypernumbers (and perhaps of other compound hypernumbers in addition). An elemental hypernumber is one of the ten basic hypernumber forms of unity ($r_1, \ldots, i_1, \ldots, \varepsilon_1, \ldots, w, p, q, \sigma, m, \Omega, v$), or a multiplicity counted in units of one of these ($5r_1$, $7i_1$, 3ε, $2w$, etc.). Geometrically, it is a point on one of the axes of $\mathbb{H}$. It is a point in $\mathbb{H}$, all of whose coordinates are zero save one.* A compound hypernumber is an additive collection of more than one qualitatively distinct hypernumber unity or of multiples thereof:

$$(r_1 + i_1 + \varepsilon_1 + w + p + q + \sigma + m + \Omega + v)$$

$$(5r_1 + 3i_1 + 6\varepsilon_1 + 2w + 7p + 9q + 4\sigma + 19m + 34\Omega + 6v).$$

It is a hypernumber "complex" or "multiplex", of which a "Complex number" is a special case, having all coordinates zero except two:

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*(See p. 98 above.)

$$a + bi \equiv ar_1 + bi_1 + 0\varepsilon_1 + 0w + 0p + 0q + 0\sigma + 0m + 0\Omega + 0v.$$

In summary, a collective operator is an inhomogeneous[69] multinomial expression constituted as an operator in its own right. It is a parenthesis containing a multiplicity of qualitatively distinct operators, the parenthesis signifying their unison for operational purposes. The addition sign connecting the terms of this multinomial effectively "brackets" them together into a single operator. That sign indicates their unitation. In both of the inscriptions below:

$$i_1 \qquad \varepsilon_1$$

$$i_1, \qquad \varepsilon_1$$

we have simply the distinct operators, i_1 and ε_1. But in

$$(i_1, \varepsilon_1) \quad \text{or} \quad \begin{bmatrix} i_1 \\ \varepsilon_1 \end{bmatrix} \quad \text{or} \quad i_1 + \varepsilon_1 \quad \text{or} \quad (i_1 + \varepsilon_1)$$

we have a single operator including both i_1 and ε_1. We can gain insight here by considering the three fashions after which a point in $\mathbb{R}^3$ can be denoted within current notational usages:

$$3\hat{x} + 5\hat{y} + 2\hat{z} \equiv (3, 5, 2) \equiv \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

In the first (leftmost) or "unit-vector" approach, the "vector unities" $\hat{x}$, $\hat{y}$ and $\hat{z}$ of the three distinct Real axes are explicitly noted. Therefore the sum form can be used because the inhomogeneity, blocking amalgamation, is evident. Otherwise, dropping the axial "tags" or "qualifiers", we would have just 3 + 5 + 2, readily mistaken for an amalgamatable expression involving terms all of which refer implicitly to points along a single Real axis, and hence reducible,to 10 -- viz. $3 + 5 + 2 \equiv 3\hat{x} + 5\hat{x} + 2\hat{x} = 10\hat{x}$.

The second approach is the "ordered n-tuple" or "coordinate-packet" approach. It uses commas, and a prescribed, fixed order of coordinate entries to bar internal amalgamation, and parentheses to indicate fusion into a whole, denoting a single point in $\mathbb{R}^n$.

Third is the "matrix" or "array" approach. The three coordinates are simply listed, again in prescribed order, and bracketed to indicate unitation, forming a one-column matrix.

The advantage of the inhomogeneous sum approach is that it obviates the need for a fixed ordering of the coordinates:

$$2\hat{z} + 5\hat{y} + 3\hat{x} = 5\hat{y} + 3\hat{x} + 2\hat{z} = 3\hat{x} + 5\hat{y} + 2\hat{z}$$

$$a + bi = bi + a.$$

The information traditionally contained in the ordering of coordinate entries is instead given independently by the unit-vector "tags", regardless of the order of occurrence. Similarly, points in $\mathbb{H}$ can be designated arithmetically by inhomogeneous sums. Addition remains commutative in the generalization to hypernumber space. Thus, the hypernumber unit "tags" or qualifier "suffixes" make order of occurrence immaterial in specifying a point. The "ordered n-tuples" of $\mathbb{R}^n$ give way to "unordered n-tuples" or "unordered n-nomials" when we come to the analytic geometry of $\mathbb{H}$. The '+' sign acts like the parentheses and comma in n-tuple notation, grouping different hypernumber "coordinates" together into a unified operational entity. The '+' sign is thus the explicit symbol of nonjuxtaposition, nonamalgamation, and operational noninteraction. It symbolizes the capacity for combining distinct operations, suspending interaction at the immediate level to allow for a new kind of operator to emerge at the level of the compound whole. Those special but more familiar instances where the sum does amalgamate, because the terms connected by the + sign are of the same quality, can be grasped as "degenerate" cases of collective operators. It is our exclusive familiarity with this degenerate case, in the form of an arithmetic confined entirely to one instance of the r axis, that makes this definition of addition seem novel.

Thus in the process arithmetic + is no longer an operation in its own right, alongside "multiplication" (or juxtaposition), but is rather a not-operation, in much the same sense that we have characterized 1 as a not-operation or "non-operator".* In fact, if we rewrite the form of a sum in "Polish notation", that is, if we treat '+' itself as an operator or function, then the similarity of '+' in function to 1 or r, and to parentheses and comma, is brought into view:

$$a + b \equiv +(a,b) \equiv +1(a,b) = 1(a,b) = r(a,b) = (ra,b) = (a,rb) = (a,b),$$

$$\text{i.e.} \quad \text{'+'} \overset{\sim}{=} (\; , \;)$$

George Boole[70] was one of the first to mention this similarity between + and 1 or r, in his The Mathematical Analysis of Logic, where he remarks upon the analogy between a fundamental formula of his arithmetic of logic:

$$1^n = 1$$

and a conception which he symbolized as follows:

$$+^n = +.$$

"Addition" thus clarified relative to the notion of an operatorial ideography as embodied in the process arithmetic, the question of "subtraction"

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[*] [See PO & LC, pp. 37-46]

naturally arises next. With subtraction, the principle is the same as with addition. The minus sign simply indicates a reversal of direction along an axis in $\mathbb{H}$. As we have hinted in previous examples, in hyperarithmetic too subtraction is the addition of the "negative" of the operator being "subtracted".* The "negative" of any operator can be grasped as the product of the operation of -1 or -r upon that operator. Thus, the '-' sign becomes identified with an "about face" or "180° rotation" with the origin-point as pivot. Subtraction is associated with the -1 operation just as 1 or +1 is associated with addition:

$$5 - 4 = 5 + (-4)$$

$$6\varepsilon + w - 4i = +6\varepsilon + w + (-4i).$$

On each hypernumber axis as on the axis of ordinary (r) numbers, there are two wings, on either side of the zero-point, and two distinct, opposed directions of extension. The points +4i and -4i are both 4 unit lengths away from the origin on the i axis. But the 4 units are counted out in opposite directions. Alternatively, -4i can be viewed as the result of rotating the "arrow" (radius vector) with tail at 0 and tip at +4i through 180° of arc counterclockwise about the point 0. The tip of the arrow will then be at -4i. Likewise, +4i can be viewed as the result of a 180° counterclockwise circular rotation from -4i. This 180° circular counterclockwise rotation is the movement associated with the operation -1 or i^2. Thus, a multinomial hypernumber expression involving "subtractions" is still an inhomogeneous sum, a collective operator, but one involving "negatively directed" hypernumber operator coordinates.

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*(In the case of the hypernumber w, however, +(-w) and -(+w) have distinct meanings, in "multiplicative" contexts, because of the dual power-orbits of w).

It remains to clarify "division". Even among ordinary numbers, the "four basic operations of arithmetic" readily reduce to one, namely, to addition. Multiplication resolves into repeated addition of the multiplicand, and can thus be regarded as shorthand for a sum:

$$ab = \underbrace{b+\ldots+b}_{a \text{ times}}$$

Division in the $\mathbb{R}^1$ context resolves into repeated subtraction of divisor from dividend, the "quotient" being a count of how many times this subtraction can occur still leaving a zero or positive remainder. Since subtraction resolves into addition of a negative number, we are left with addition as the primary operation of arithmetic. This theoretical point becomes practical in the construction of the binary-based digital computer.

More reminiscent of our general operatorial approach is the reformulation of division among ordinary numbers as a form of multiplication. The division of b by a can be expressed as the product of b with a^{-1}, the "multiplicative-" inverse operator to $a \equiv a^{+1}$. Equivalently, $b \div a$ can be expressed as the product of b with the reciprocal of a:

$$b \div a \rightarrow c$$

$$b \div a \equiv a^{-1}b \equiv \frac{1}{a}(b) \rightarrow c$$

Simply renaming a^{-1} or $\frac{1}{a}$ to effect a direct symbolization -- $a^{-1} \equiv d \equiv \frac{1}{a}$ -- brings the form $(b \div a) \rightarrow c$ back within our general "flexion" paradigm:

$$db \rightarrow c$$

However, neither the identification of division with subtraction (ultimately with addition) nor of inverse[*] operators with reciprocals[†] of operators survives the generalization to $\mathbb{H}$. Both identities break down in hypernumber contexts. For example, the first negative power of the hypernumber integral operator σ is the hypernumber differential operator[**]:

$$\sigma^{-1}() = \frac{d()}{dt} s(t)$$

The reciprocal of σ is quite another matter[71]:

$$\frac{1}{\sigma()} = \frac{1}{\int () s(t) dt}$$

Also, $(1+\varepsilon)^{-1} = \frac{(1+\varepsilon)}{4}$ and $(1-\varepsilon)^{-1} = \frac{(1-\varepsilon)}{4}$.[72] That is,

$$(1\pm\varepsilon)^{-1} \neq \frac{1}{(1\pm\varepsilon)} .$$

This breakdown occurs because the identifications above depend crucially upon the character of ordinary numbers as "countors" or pure cardinalities. The "Natural" numbers represent the operation of counting. Hypernumbers represent other operations with quite different properties. Thus, in the generalization to $\mathbb{H}$ as a whole, $\mathbb{R}^1$ "division" divides into at least two distinct branches: (i) flexion by the negative powers of an operator, that is, by the inverse* operation; and (ii) flexion by the reciprocal of an operator:

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[*] (the "negative-first power" of an operator)

[†] ("one over" an operator, "one" meaning Real unity, $r \equiv 1$)

[**] [s(t) symbolizes the Heaviside unit step function. See p. 67n, above.]

$$\frac{1}{a}(b) \neq a^{-1}(b).$$

We have thus established that any of the forms below

$$ab \;;\; a(b)$$
$$(a + b + c + \ldots + z)$$
$$(-a - b - c - \ldots - z)$$
$$(a - b + c - d + \ldots)$$
$$a \div b \;;\; b^{-1}(a) \;;\; \frac{1}{b}(a)$$
$$(a + b + \ldots)c$$
$$(a + b + \ldots)\bullet(x + y + \ldots)$$

are not only meaningful in the process arithmetic, but refer to specific points in $\mathbb{H}$. When some or all of the axes of $\mathbb{H}$ are assigned to specific state-variables of a dynamical system, thus interpreting $\mathbb{H}$ as a state-space, each of the above therefore denotes also a definite state of the dynamical system. Each expression above is an operation-valued expression. Each expression "reduces" to a definite hypernumber operator. And every hypernumber operator is associated with a definite, unique point in $\mathbb{H}$, the space of operators, as the name of that point.

$\mathbb{H}$ versus $\mathbb{R}^n$: Concluding Observations

We conclude this exposition of the "process-algebra" concept with a sequence of eleven summary observations about hypernumber space as a state-space and about the meaning of hypernumber reflexive functions.

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*(The inverse of an operation a is the "antidote" to a; the operation that undoes what a does, so that the product of a with its inverse is the identity operation, i.e. "no change": $a^{+1}a^{-1} = a^{0} = r \equiv 1$. In cases of hypernumbers which are nilpotent this breaks down. In those cases, the zeroth power may be a zero: $a^{0} = 0$.[73])

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$$6r + (-3r) = 3r$$

$$5i_1 + 10r - 3i_1 + 6r = 16r + 2i_1 .$$

Note, however, that every such hyperarithmetic expression, homogeneous or inhomogeneous, amalgamative or not, refers to or "picks out" a definite, unique point in $\mathbb{H}$, and therefore a definite, unique system state, whenever $\mathbb{H}$ has been interpreted as a state space.

Besides Complex arithmetic,* the deployment of unit vectors in Real vector arithmetic provides our most familiar precedent for calculative procedures similar to those exemplified above. In $\mathbb{R}^3$, for example, we often denote the unit vector of the first Real axis by the symbol $\hat{x}$ ($\equiv r_1$ in the hypernumber notation), that of the second by $\hat{y}$ ($\equiv r_2$), and that of the third by $\hat{z}$ ($\equiv r_3$).†

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*(which is actually a subcase of hyperarithmetic, belonging to a subspace of $\mathbb{H}$.)

†(See p. 99 above.)

We then represent points* in $\mathbb{R}^3$ by nonamalgamative, inhomogeneous sums like the following:

$$3\hat{x} + 2\hat{y} + 5\hat{z}.$$

However, in the special case of multiplicities qualified by the same unit vector "tag" -- that is, in the case of homogeneous sums -- amalgamation does occur:

$$3\hat{x} + 6\hat{x} + (-2\hat{x}) = 7\hat{x}$$

$$6\hat{x} + 9\hat{y} + 3\hat{x} + \hat{x} + 2\hat{z} = 10\hat{x} + 9\hat{y} + 2\hat{z}.$$

Overall, the best way to conceive "addition" in its generalized, hyperarithmetic meaning, is as the formation of a 'collective operator', and as the indication of noninteraction among the individual operators which are the internal components of that collective operator. By a collective operator, we mean an array or sum of qualitatively distinct operators which are treated as a composite whole. That is, they operate as a single entity, a single operator, upon other operators. The collective operator acts as a united whole externally, but its component operators do not interact among themselves internally. Complex arithmetic provides the most accessible examples of 'collective operators'. For instance, the Complex number 5 + 3i ($5r_1 + 3i_1$ in hypernumber notation) is a collective operator, representing a definite point in the Complex plane. When 5 + 3i interacts with another Complex number, say 2 + 4i, it is applied to that number as a composite whole:

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*(that is, vectors -- often depicted geometrically as "arrows" pointing to or pointing out said points.)

$$\begin{aligned}(5 + 3i)(2 + 4i) &= 5(2) + 5(4i) + 3i(2) + 3i(4i)\\ &= 10 + 20i + 6i + 12i^2\\ &= (10 - 12) + (20 + 6)i = -2 + 26i \ (= -2r_1 + 26i_1).\end{aligned}$$

But inside the parenthesis which encloses 5 and 3i, collecting them together into a single, "compound" operator, there is no interaction. The 5 does not operate upon the 3i, nor vice versa. Nor does the 5 alone, or the 3i alone, operate when an external operator such as (2 + 4i) is juxtaposed. Both the 5 and the 3i act upon (2 + 4i). And they act upon the whole of it, not just upon 2 or 4i. The 5 and the 3i act together, in unison, as if a single entity. The "+" sign binds them together. That is, it signifies they are to be treated as one entity for purposes of calculation.

In the process arithmetic, a collective operator is a "compound" hypernumber. A compound hypernumber is made up of "elemental"[68] hypernumbers (and perhaps of other compound hypernumbers in addition). An elemental hypernumber is one of the ten basic hypernumber forms of unity (r_1, ..., i_1, ..., ε_1, ..., w, p, q, σ, m, Ω, v), or a multiplicity counted in units of one of these ($5r_1$, $7i_1$, 3ε, 2w, etc.). Geometrically, it is a point on one of the axes of $\mathbb{H}$. It is a point in $\mathbb{H}$, all of whose coordinates are zero save one.*
A compound hypernumber is an additive collection of more than one qualitatively distinct hypernumber unity or of multiples thereof:

$$(r_1 + i_1 + \varepsilon_1 + w + p + q + \sigma + m + \Omega + v)$$

$$(5r_1 + 3i_1 + 6\varepsilon_1 + 2w + 7p + 9q + 4\sigma + 19m + 34\Omega + 6v).$$

It is a hypernumber "complex" or "multiplex", of which a "Complex number" is a special case, having all coordinates zero except two:

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*(See p. 98 above.)

$$a + bi \equiv ar_1 + bi_1 + 0\varepsilon_1 + 0w + 0p + 0q + 0\sigma + 0m + 0\Omega + 0v.$$

In summary, a collective operator is an inhomogeneous[69] multinomial expression constituted as an operator in its own right. It is a parenthesis containing a multiplicity of qualitatively distinct operators, the parenthesis signifying their unison for operational purposes. The addition sign connecting the terms of this multinomial effectively "brackets" them together into a single operator. That sign indicates their unitation. In both of the inscriptions below:

$$i_1 \qquad \varepsilon_1$$

$$i_1, \qquad \varepsilon_1$$

we have simply the distinct operators, i_1 and ε_1. But in

$$(i_1, \varepsilon_1) \quad \text{or} \quad \begin{bmatrix} i_1 \\ \varepsilon_1 \end{bmatrix} \quad \text{or} \quad i_1 + \varepsilon_1 \quad \text{or} \quad (i_1 + \varepsilon_1)$$

we have a single operator including both i_1 and ε_1. We can gain insight here by considering the three fashions after which a point in $\mathbb{R}^3$ can be denoted within current notational usages:

$$3\hat{x} + 5\hat{y} + 2\hat{z} \equiv (3, 5, 2) \equiv \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$$

In the first (leftmost) or "unit-vector" approach, the "vector unities" $\hat{x}$, $\hat{y}$ and $\hat{z}$ of the three distinct Real axes are explicitly noted. Therefore the sum form can be used because the inhomogeneity, blocking amalgamation, is evident. Otherwise, dropping the axial "tags" or "qualifiers", we would have just 3 + 5 + 2, readily mistaken for an amalgamatable expression involving terms all of which refer implicitly to points along a single Real axis, and hence reducible, to 10 -- viz. $3 + 5 + 2 \equiv 3\hat{x} + 5\hat{x} + 2\hat{x} = 10\hat{x}$.

The second approach is the "ordered n-tuple" or "coordinate-packet" approach. It uses commas, and a prescribed, fixed order of coordinate entries to bar internal amalgamation, and parentheses to indicate fusion into a whole, denoting a single point in $\mathbb{R}^n$.

Third is the "matrix" or "array" approach. The three coordinates are simply listed, again in prescribed order, and bracketed to indicate unitation, forming a one-column matrix.

The advantage of the inhomogeneous sum approach is that it obviates the need for a fixed ordering of the coordinates:

$$2\hat{z} + 5\hat{y} + 3\hat{x} = 5\hat{y} + 3\hat{x} + 2\hat{z} = 3\hat{x} + 5\hat{y} + 2\hat{z}$$

$$a + bi = bi + a.$$

The information traditionally contained in the ordering of coordinate entries is instead given independently by the unit-vector "tags", regardless of the order of occurrence. Similarly, points in $\mathbb{H}$ can be designated arithmetically by inhomogeneous sums. Addition remains commutative in the generalization to hypernumber space. Thus, the hypernumber unit "tags" or qualifier "suffixes" make order of occurrence immaterial in specifying a point. The "ordered n-tuples" of $\mathbb{R}^n$ give way to "unordered n-tuples" or "unordered n-nomials" when we come to the analytic geometry of $\mathbb{H}$. The '+' sign acts like the parentheses and comma in n-tuple notation, grouping different hypernumber "coordinates" together into a unified operational entity. The '+' sign is thus the explicit symbol of nonjuxtaposition, nonamalgamation, and operational noninteraction. It symbolizes the capacity for combining distinct operations, suspending interaction at the immediate level to allow for a new kind of operator to emerge at the level of the compound whole. Those special but more familiar instances where the sum does amalgamate, because the terms connected by the + sign are of the same quality, can be grasped as "degenerate" cases of collective operators. It is our exclusive familiarity with this degenerate case, in the form of an arithmetic confined entirely to one instance of the r axis, that makes this definition of addition seem novel.

Thus in the process arithmetic + is no longer an operation in its own right, alongside "multiplication" (or juxtaposition), but is rather a not-operation, in much the same sense that we have characterized 1 as a not-operation or "non-operator".* In fact, if we rewrite the form of a sum in "Polish notation", that is, if we treat '+' itself as an operator or function, then the similarity of '+' in function to 1 or r, and to parentheses and comma, is brought into view:

$$a + b \equiv +(a,b) \equiv +1(a,b) = 1(a,b) = r(a,b) = (ra,b) = (a,rb) = (a,b),$$

$$\text{i.e.} \quad \text{'+'} \overset{\sim}{=} (\ ,\ )$$

George Boole[70] was one of the first to mention this similarity between + and 1 or r, in his The Mathematical Analysis of Logic, where he remarks upon the analogy between a fundamental formula of his arithmetic of logic:

$$1^n = 1$$

and a conception which he symbolized as follows:

$$+^n = +.$$

"Addition" thus clarified relative to the notion of an operatorial ideography as embodied in the process arithmetic, the question of "subtraction" naturally arises next. With subtraction, the principle is the same as with addition. The minus sign simply indicates a reversal of direction along an axis in $\mathbb{H}$. As we have hinted in previous examples, in hyperarithmetic too subtraction is the addition of the "negative" of the operator being "subtracted".* The "negative" of any operator can be grasped as the product of the operation of -1 or -r upon that operator. Thus, the'-'sign becomes identified with an "about face" or "180° rotation" with the origin-point as pivot. Subtraction is associated with the -1 operation just as 1 or +1 is associated with addition:

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*[See PO & LC, pp. 37-46]

$$5 - 4 = 5 + (-4)$$

$$6\varepsilon + w - 4i = +6\varepsilon + w + (-4i).$$

On each hypernumber axis as on the axis of ordinary (r) numbers, there are two wings, on either side of the zero-point, and two distinct, opposed directions of extension. The points +4i and -4i are both 4 unit lengths away from the origin on the i axis. But the 4 units are counted out in opposite directions. Alternatively, -4i can be viewed as the result of rotating the "arrow" (radius vector) with tail at 0 and tip at +4i through 180° of arc counterclockwise about the point 0. The tip of the arrow will then be at -4i. Likewise, +4i can be viewed as the result of a 180° counterclockwise circular rotation from -4i. This 180° circular counterclockwise rotation is the movement associated with the operation -1 or i^2. Thus, a multinomial hypernumber expression involving "subtractions" is still an inhomogeneous sum, a collective operator, but one involving "negatively directed" hypernumber operator coordinates.

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*(In the case of the hypernumber w, however, +(-w) and -(+w) have distinct meanings, in "multiplicative" contexts, because of the dual power-orbits of w).

It remains to clarify "division". Even among ordinary numbers, the "four basic operations of arithmetic" readily reduce to one, namely, to addition. Multiplication resolves into repeated addition of the multiplicand, and can thus be regarded as shorthand for a sum:

$$ab = \underbrace{b+\ldots+b}_{a \text{ times}}$$

Division in the $\mathbb{R}^1$ context resolves into repeated subtraction of divisor from dividend, the "quotient" being a count of how many times this subtraction can occur still leaving a zero or positive remainder. Since subtraction resolves into addition of a negative number, we are left with addition as the primary operation of arithmetic. This theoretical point becomes practical in the construction of the binary-based digital computer.

More reminiscent of our general operatorial approach is the reformulation of division among ordinary numbers as a form of multiplication. The division of b by a can be expressed as the product of b with a^{-1}, the "multiplicative-" inverse operator to $a \equiv a^{+1}$. Equivalently, $b \div a$ can be expressed as the product of b with the reciprocal of a:

$$b \div a \rightarrow c$$

$$b \div a \equiv a^{-1}b \equiv \frac{1}{a}(b) \rightarrow c$$

Simply renaming a^{-1} or $\frac{1}{a}$ to effect a direct symbolization -- $a^{-1} \equiv d \equiv \frac{1}{a}$ -- brings the form $(b \div a) \rightarrow c$ back within our general "flexion" paradigm:

$$db \rightarrow c$$

However, neither the identification of division with subtraction (ultimately with addition) nor of inverse* operators with reciprocals† of operators survives the generalization to $\mathbb{H}$. Both identities break down in hypernumber contexts. For example, the first negative power of the hypernumber integral operator σ is the hypernumber differential operator**:

$$\sigma^{-1}() = \frac{d()}{dt}\, s(t)$$

The reciprocal of σ is quite another matter[71]:

$$\frac{1}{\sigma()} = \frac{1}{\int () s(t) dt}$$

Also, $(1+\varepsilon)^{-1} = \frac{(1+\varepsilon)}{4}$ and $(1-\varepsilon)^{-1} = \frac{(1-\varepsilon)}{4}$.[72] That is,

$$(1\pm\varepsilon)^{-1} \neq \frac{1}{(1\pm\varepsilon)} .$$

This breakdown occurs because the identifications above depend crucially upon the character of ordinary numbers as "countors" or pure cardinalities. The "Natural" numbers represent the operation of counting. Hypernumbers represent other operations with quite different properties. Thus, in the generalization to $\mathbb{H}$ as a whole, $\mathbb{R}^1$ "division" divides into at least two distinct branches: (i) flexion by the negative powers of an operator, that is, by the

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*(the "negative-first power" of an operator)

†("one over" an operator, "one" meaning Real unity, $r \equiv 1$)

**[s(t) symbolizes the Heaviside unit step function. See p. 67n, above.]

inverse[*] operation; and (ii) flexion by the *reciprocal* of an operator:

$$\frac{1}{a}(b) \neq a^{-1}(b).$$

We have thus established that any of the forms below

$$\begin{array}{l}
ab \;\; ; \;\; a(b) \\
(a + b + c + \ldots + z) \\
(-a - b - c - \ldots - z) \\
(a - b + c - d + \ldots\ ) \\
a \div b \; ; \;\; b^{-1}(a) \;\; ; \;\; \frac{1}{b}(a) \\
(a + b + \ldots)c \\
(a + b + \ldots)\bullet(x + y + \ldots)
\end{array}$$

are not only meaningful in the process arithmetic, but refer to specific *points* in $\mathbb{H}$. When some or all of the axes of $\mathbb{H}$ are assigned to specific state-variables of a dynamical system, thus interpreting $\mathbb{H}$ as a state-space, each of the above therefore denotes also a definite *state* of the dynamical system. Each expression above is an *operation-valued* expression. Each expression "reduces" to a definite hypernumber operator. And every hypernumber operator is associated with a definite, unique point in $\mathbb{H}$, the space of operators, as the *name* of that point.

$\mathbb{H}$ versus $\mathbb{R}^n$: Concluding Observations

We conclude this exposition of the "process-algebra" concept with a sequence of eleven summary observations about hypernumber space as a state-space and about the meaning of hypernumber reflexive functions.

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[*] (The inverse of an operation a is the "antidote" to a; the operation that *undoes* what a does, so that the product of a with its inverse is the identity operation, i.e. "no change": $a^{+1}a^{-1} = a^{0} = r \equiv 1$. In cases of hypernumbers which are *nilpotent* this breaks down. In those cases, the zeroth power may be a zero: $a^{0} = 0$.[73])

1. Operators 'On' and 'In' a Space

The crucial feature of $\mathbb{H}$ is that operations on it cease to be "outside agitators" as they are in the case of $\mathbb{R}^n$. $\mathbb{H}$ contains its operators. They are internal to the space. Operators "on" the space $\mathbb{H}$ also name points inside it. This enables $\mathbb{H}$ to represent a "space of process" and to depict reflexive functions and other generalized "multiplications" (flexions) as trajectories within it.*

In $y = f(x)$, we are used to thinking of f as belonging to a radically different order of mathematical entity than the x (or the y). The x and the y denote "values", "numbers", "quantities". Analytic-geometrically, they denote points in Cartesian-Euclidean Space, $\mathbb{R}^2$. The expression $f(x)$ or "fx" is called a "function-value". The f by itself, on the other hand, is ordinarily not considered a "value" at all. At most, it is conceived as a potential value -- as something like a "catalyst". It will turn one value (x) into another (y) once it "comes into contact" with numbers (values). It will turn out numbers once it gets next to an argument "x". The f is not regarded as representable by a point in the same space where x and y are. At best, the f is conceived as a point in an "abstract" (undepictable) function-space. But this space is conceived as always "elsewhere" with respect to the space of x and y.

In an operatorial ideography, however, y, f and x all belong to the same syntactical and 'ideogrammatical' class. Operand is always also operator. The argument too is a function. The f in $y = fx$, the a in $c = ab$ $(ab \rightarrow c)$ are already also (hyper)values, (hyper)numbers, generalized "quantities". The f is representable by a point along the f axis in $\mathbb{H}$. All the permuted forms, $z = x(f)$; $x = x(y)$; $u = y(f)$; and $f = r(s)$, make equal syntactic and ideogrammatical sense.

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[*] [That is, in this operatorial space, a function appears as a movement from the point denoting the "argument" to the point denoting the function-value for that argument or operand. This contrasts with the more familiar notion of a function as a static mapping or passive "association" from one set (space) to another.]

The conjectured Fundamental Theorem of Nonlinear Algebra requires, among other conditions, that $\mathbb{H}$ be "closed" operationally, that is, closed with respect to all flexions among operators it contains. In ideographic symbols, this condition is expressed by*: $(\forall u)\ u \,\varepsilon\, \mathbb{H} \rightarrow u: \mathbb{H} \rightarrow \mathbb{H}$. In words: For every hypernumber u in $\mathbb{H}$, u maps $\mathbb{H}$ back into itself, that is, every point in $\mathbb{H}$ is removed to some other point in $\mathbb{H}$ by the operation u, along a trajectory that remains entirely within $\mathbb{H}$ throughout the transition.

2. Fields, Intrinsic and Extrinsic

A "vector field" on $\mathbb{R}^n$ is defined as a function which assigns a vector to each point of $\mathbb{R}^n$. The form of the "flow" indicated by this assignment is considered quite arbitrary, an extrinsic matter depending entirely upon the vector field function chosen, without reference to or constraint by any properties intrinsic to $\mathbb{R}^n$ itself. $\mathbb{R}^n$ is "flow-neutral". It can "take" any conceivable vector field, and any conceivable vector field can be defined on it. With hypernumber spaces, subspaces of $\mathbb{H}$, on the contrary, certain field-patterns are "privileged". They are intrinsic to the operational quality forming the subspace. It is as if the various subplanes of $\mathbb{H}$ had congenital "grooves"; a natural "grain" of their own. Thus, the $r \perp w$ plane has an elliptical grain; the $r \perp i$ plane a circular grain; the $p \perp q$ plane a lobate grain, etc. Hypernumber space in effect selects certain field patterns and separates them, allocating them to specific, established subspaces within itself.

These seven distinct field patterns must, if the conjectured "Fundamental Theorem" be true, be fundamental or "primary" with respect to all others,[73.5] in some sense yet to be fully explicated. Perhaps, like the television camera's three primary color filters, blue, yellow, and red, by blending whose images all others can be expressed, these seven power-orbit patterns suffice to describe any other trajectory-pattern, any other "field".

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* (Here "ε" represents the set-inclusion relation, not the hypernumber unity.)

Different types of state space trajectory, hence different types of systems, are assigned to different subplanes (or other, more than two-dimensional, sub*spaces*) of $\mathbb{H}$. That is, they are assigned to different "intrinsic field" *regions* of $\mathbb{H}$.

This concept of intrinsic flows leads us to consider the concept of the mathematical *point* in a new light. Rather than as an isolated and independent entity in its own right, we might regard a point in any subspace of $\mathbb{H}$ as merely a *moment of a flow* -- of *the* flow intrinsic to that subspace.

3. *The Conceptual Meaning of Hyperfunctions*

A hyperfunction model $x(t) = g(t) \cdot u^{f(t)}$ or $y(t) = g(t) \cdot e^{\theta u f(t)}$ describes the *reflexive action* of a system X. In the simplest case, we have $x(t) = a^{2^t}$. This says that, in the beginning, at $t = 0$, the state of the process or eventity X is "measured" by the hypernumber value a: $x(0) = a$. By $t = 1$, however, this state of activity has once fully acted upon itself, producing the state $a(a)$ or a^2: $x(1) = a^2$. That is, by $t = 1$, the state of the system is no longer "measured" by the value "*a*". The "measure" of the state of the system has become a^2. The "output" of the system is now the activity a^2, no longer *a*. Then, by $t = 2$, this new state has fully acted upon or reflected upon itself, resulting in the state $a^2(a^2)$ or a^4. By time $t=1$, the activity put out is no longer a^1, so the "echo" or reflux of that activity re-input into the system, the self-effect of that activity, is not $a^1(a^2) = a^3$ but $a^2(a^2) = a^4$. This is the significance of the exponent-function's form 2^t or, more generally, $2^{f(t)}$. That function keeps the reflection current, so that what reflects upon the current state (activity) is the current state (activity) of the system, rather than a past state whose effect has already become fully registered and embodied previously, i.e. at earlier values of t. Thus the reflexive function $x(t) = a^{2^{f(t)}}$ depicts the continuous self-activity of a system, a situation wherein the system-state interacts with itself continuously and recursively (recurrently) in determining the development or dynamical evolution of the system-state. The reflexive function form is thus a mathematical model of *reflexive process* in general, and of the concept of "self-interaction" of system-states into which we resolved the meaning of "nonlinearity" in integrodifferential equations. For hypernumber values of *a*, every non-integral value of the exponent t or f(t), including irrational as well as fractional values (roots), represents a *partial* self-action of the system-state, an intermediate stage in the completion of the reflexion or self-interaction. The function-values $a^{2^{f(t)}}$ represent the power-*continuum* of the operator *a*, that is, the *continuum of its Real powers*. With suitable "principal value" arrangements, this power-continuum depicts geometrically as a continuous trajectory in $\mathbb{H}$.

4. *Modeling Evolutionary Acceleration*

The form of the exponent, or "frequency" (of reflexion) function, $2^{f(t)}$, accommodates description in the hyperfunction form of an important feature of observed behavior in evolution-processes. This feature is evolutionary acceleration, or "acceleratory development". Recently a number of scientists have designated this as a Law of Nature.[74] The acceleration of social evolution, in terms of basic technologies of social livelihood, and of economic forms, is today a commonplace perception. Scientists have noted the acceleratory pattern of cosmic evolution as a whole. That is, the atom appears to have been the product of a labor of much longer duration in the womb of Nature than it took to produce, after the atom was achieved, the molecule -- the "atom of atoms" or "atom of second degree" ("atom^2"). The molecule took longer than it took, after arriving at the molecule, to produce from it the cell, the "molecule of molecules". The multicellular organism, or "cell of cells", was achieved, building upon the basis of the cell, much more quickly than was the cell itself, from the basis of the molecule. Finally, societies, or "organism of (multicellular) organisms", developing from multicellular organisms, took a fraction of the time it took multicellular organisms to develop from single-celled organisms.

In classical mechanics, in terms of _spatial_ motion, it is _acceleration_, not mere velocity, however great, which signifies the presence of "force".[75] It appears that the celerity of motion in general -- not restricted to mechanical (spatial) motion, but including the generalized "motion" of growth and evolution as well -- has a significance, in general systems theory, analogous -- or should we say homologous? -- to that of spatial acceleration in Newtonian systems. Via the state-space metaphor, the acceleration of general processes can be depicted as an acceleration of spatial motion -- namely, as accelerated movement of the point representing the state of the system (process) as it traces out the state-space trajectory. The hyperfunction form $\underline{x}(t) = g(t)u^{2^{f(t)}}$ expresses evolutionary acceleration wherever $f(t)$ is a monotonically increasing nonlinear function of time, such as $f(t) = t^4$ or $f(t) = t^7$, etc., with _increasing_ slope [which preserve $f(0) = 0$; $\underline{x}(0) = g(0)u^1$ and $f(1) = 1$; $\underline{x}(1) = g(1)u^2$] :

$$\underline{x}(t) = g(t)u^{2^{t^4}} \quad \text{or} \quad \underline{x}(t) = g(t)u^{2^{t^7}}$$

Such function-forms, involving three different layers of "nested" exponentiation, appear rather exotic, having hardly been looked at in the past.[76]

But it is only to be expected that the "missing functions" needed to solve nonlinear integrodifferential equations should exhibit a novel, or even cumbersome appearance in terms of notations developed prior to their recognition.

5. Hyperfunction Modeling of "Creative Processes"

So far, our reflexive function form seems afflicted with an undue degree of determinism. All future states of a self-interacting, "self-evolving" system modeled by such a state-function appear perfectly pre-destined by the value of u, or a; that is, by the value of the hypernumber constant which centers the reflexive part of the hyperfunction.[†]

But suppose that central operator u is not a constant. Suppose it itself is a function of time: U(t). Then our hyperfunction form acquires no less than three independent loci of time-variation, the first in g(t), the second in $2^{f(t)}$, and the third in u = U(t) itself. (i) As discussed earlier, the time-function g(t) accounts current action of states of the system's environment upon the system which are independent of any influence of the system's own state upon that environment. (ii) The time-function $2^{f(t)}$ describes the pacing of self-interaction of the system's own states, the rhythm of reflexion, the timing of the self-effects and self-consequences of each system-state. (iii) The hypernumber-valued time-function $u \equiv U(t)$ can accommodate the description of "creative" processes within the hyperfunction form, whose "values" may be unknown until after the fact.

By "creative processes" I mean processes which give rise to novelty; and processes whose outcomes may be unpredictable, unexpected, unprecedented, singular, or unique. "Creative processes" would include first-time evolutionary events giving birth to new qualities in Nature, up to and including those acts of creative choice and initiative by human individuals and collectivities traditionally comprehended under the category of "free will". This generalized hyperfunction form

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[†] [Apart from environmental vicissitudes covered by g(t).]

$$\underline{x}(t) = g(t)[U(t)]^{2^{f(t)}}$$

can also accommodate processes whose values are unknown in practice due to insufficient information, as well as in principle, due to the freedom and creativity, or inherent unpredictability, of the agent of action, human or otherwise.

In general, U(t) may be a complicated "sum" of hypernumber-valued functions:

$$U(t) = a(t)\underline{r} + b(t)\underline{i} + c(t)\underline{c} + d(t)\underline{w} + e(t)\underline{p} + h(t)\underline{q} + j(t)\underline{m} + k(t)\underline{\Omega} + \ell(t)\underline{v} \ldots$$

or an "array" ("collective operator") of such hypernumber-valued functions of time. Sufficiently dramatic changes in the value of, or values and value-structure within, U(t), may require an alteration in f(t), that is, may alter the frequency of self-reflexion. Thus, it would be more general to rewrite $2^{f(t)}$ as a function of (an operation upon) U(t), not just of (upon) t alone.

$$\underline{x}(t) = g(t)[U(t)]^{2^{f[U(t),t]}}$$

6. The Geometry of Hyperfunctions (and the Explicit Temporalization of State-Space)

Look again at Fig. 7, p. 84. Note especially (7b) in the central, fullest depiction, the one with the time-axis explicitly shown in addition to the other two, r and w, state-variable axes. In that space, the hyperfunction $\underline{x}(t) = g(t) \cdot w^{f(t)}$ (given more explicitly just below the Figure on p. 84) graphs-out as a helix. To be exact, its graph is an irregular vortical helix. This helical trajectory is ever-narrowing in radius as it whirls around and around out along the t-axis, asymptotically converging upon a final helical pattern of fixed size and shape, an irregular variation on an elliptical-helical theme. This is because the factor $\frac{2e^t + 1}{2e^t - 1}$, beginning at t = 0 with value 3.0, asymptotically approaches the value +1 as $t \to +\infty$. The irregularity is contributed by the factor $(\cos(\sin \frac{\pi}{3} f(t))$. The latter corresponds to the factor v(t), and the former to the factor a(t), of g(t) as presented on p. 70 above.

Geometrically, the hyperfunctions in general are HELICES. The w-hyperfunctions are (variations upon) elliptical-helical; the i-hyperfunctions, $g(t)i^{f(t)}$, circular (cylindrical)-helical; the p-hyperfunctions lobate-helical; the m-hyperfunctions cassinian-oval-helical; the Ω-hyperfunctions cornu-spiral-helical; and so on. They are (in the case of attractors) asymptotically closing; vortical helices. Just such are the limit-cycles, if a time axis is admitted to the state-space. However, the time axis is normally not included in state-space. Instead the time-values associated with points on the limit-cycle-approaching state-space trajectory are dubbed in extraneously as in (7a). The graphs (7c) and (7d) are treated as belonging to spaces totally distinct from one another and from (7a) and (7b). Including a time-axis in state-space has the advantage (among others) of making Figs. (7a), (7b), (7c), and (7d) just different "side" views of a single 3-dimensional $(i \perp w) \perp t$ space (See PO & LC, p. 81).

The resulting space is timed state-space, or "state-space-time". This explicitly temporalized space is the geometric counterpart of the explicitly temporalized process-arithmetic, in which the general process-form $ab \to c$ becomes the dynamical "flow" form $s(t) = a^{f(t)} \cdot b$ (as discussed on p. 76 above). This space arises readily from the principle that the graphic locus representing geometrically the meaning of an algebraic expression must include an axis, or *other geometric counterpart,* for *every* distinct symbol in the algebraic expression. Thus, the rudimentary hyperfunction $x(t) = i^t = \underline{r}\cdot\cos\frac{\pi}{2}t + \underline{i}\cdot\sin\frac{\pi}{2}t$ is not adequately depicted by showing simply the circular i-power-orbit in the $i \perp r$ "Complex plane". An axis *is* provided for i and for r. But nothing in the picture corresponds to t. However, t occurs in the algebraic formula (as the exponent of i or as the variable factor in the arguments of the sine and cosine functions). The picture is therefore incomplete. The complete locus of i^t is the "spring"-shaped right cylindrical *helix* rising out of the $i \perp r$ plane, and winding out around the *t-axis*.

This helical geometry associated with hypernumber *dynamical* reflexive functions points up again their meaning and that of the hypernumbers themselves with respect to state-space applications. A living, dynamic system is a *fluctuating*, *vibrating* system, in terms of the measurements about its ever-changing state which constitute its state-variables. Such a system is characterized -- *fingerprinted* -- by a definite *pattern of interplay* of the fluctuations of its various measurements. That pattern of interplay captures the *dynamical essence* of the system, its essential, defining and identifying behavior; its "law of motion". The state-space trajectory is nothing but a unified pictorial symbol for this pattern of *co-related fluctuations.** The various hypernumbers and their reflexive functions provide a basic *vocabulary* of such patterns of corelated fluctuation from which to build up a description of the specific trajectory of any particular system. The i-type reflexive function, for example, represents the co-relation of a sine and cosine wave, and so on.

The inclusion of a t-axis in hypernumber state-space, that is, in $\mathbb{H}$ interpreted for use as a state-space, is quite natural for yet another reason. The hypernumber $\sigma(\ )$ is defined as a very general, powerful form of the integral operator. It is defined as a form of the Stieltjes integral, involving the unit-step function s(t):[77]

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*[See PO & LC, p. 87.]

$$\sigma^{n}(\ ) \equiv \underbrace{\int\int.....\int}_{n}(\ )s(t)dt.$$

In its pure form $\sigma \equiv \sigma(\ )$, this hypernumber unity has no special axis, but is capable of acting along every (hyper)number axis. But, just as ir, wr, εr, pr, etc. represent movements from the unit-point on the r-axis, r itself, to the unit points i, w, ε, p, etc. on the i, w, ε, p, etc. axes respectively, so $\sigma(r) = \sigma r \equiv \int(1)s(t)dt \equiv \int s(t)dt$ represents a point on a definite axis. For

$$\sigma r = \int s(t)dt = t. \quad \left\{\begin{array}{ll} ir \to i & ri \to i \\ wr \to w & rw \to w \\ \varepsilon r \to \varepsilon & r\varepsilon \to \varepsilon \\ pr \to p & rp \to p \\ & \\ \sigma r \to t & r\sigma \to \sigma \end{array}\right\}$$

The axis corresponding to the hypernumber σ is thus the t-axis, and the t-axis is therefore a normal part of hypernumber state-space.

7. The Power of A Process

In geometrizing a rudimentary hyperfunction of the form $\underline{x}(t) = u^{t}$, with u a hypernumber unit, we noted that the space formed by the u-axis plus the (one or more) other(s) necessary to depict the power-orbit of u, is insufficient. We proposed the rule that the geometry corresponding fully to such a state-formula required a geometric counterpart for every major functional symbol or symbol-complex in the formula. In particular, for the case in point, we required a t-axis corresponding to the t exponent. In more general hyperfunction forms $\underline{x}(t) = g(t)\ u^{f(t)}$ we want an f(t) axis in addition to, or in place of, the t-axis.

In such models, f(t) is typically also a state-variable. But, occurring in the state-formula at the superscript, the exponential or "meta" level, i.e. in the "power" position, f(t) has a certain "presiding" role relative to the other state-variables, which occur at the 'scriptal' level. The f(t) axis represents indeed a "metadimension" with respect to the other state-variable dimensions. In such a position, f(t) typically represents the fundamental *independent state-variable*; the causative and *controlling* state-variable, of the whole process/eventity whose state is given by $X \equiv \underline{x}(t)$. It typically measures that key subprocess which is the driving force, energizing principle, and causal dynamic; the prime mover, in short the motivating "power" of the whole evolution described by $\underline{x}(t)$. In particular, f(t) may measure the central accumulation/depletion process of the system being modeled.* It thus accounts the essential dynamic which propells the system through its various stages of development *internally*, and then, "externally", *beyond itself*, when the limits of that accumulation/depletion mode are breached (see Section II.3, "The Boundaries of the Economy", below).

This "propulsion-function" f(t) is "time-like" in the sense that it measures *concrete duration*, relative time; time *for* the system in question. It measures the process that represents maturation for that specific system. The function f(t) is monotonically increasing to the extent that (or whenever) the system is actually advancing along its natural, intrinsic path of development. The process described by f(t) is the "immanent clock" of that system-process as a whole; the self-clocking mechanism of that process. It measures time *relative to* the system; relative to the system's internal or self-development. It measures the particular local time, the TIME OF that system. The f(t) axis can thus substitute for the general t-axis in the state-space of the system. The ordinary t-axis supposedly measures abstract, absolute, universal, uniform time (actually, it usually measures earth-rotation/solar time). The state-variable function f(t), on the other hand, measures immanent, local, system-relative time. It measures the output, the actual flow, of the driving power of the system-process, in all its vicissitudinous concreteness. It measures actualized time; the time of system-evolution actually achieved, all faltering taken into account.

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*[The concept of the "central accumulation/depletion process" of a system or stage of evolution is developed in part 3, "The Boundaries of the Economy."]

8. The Principle of Comprehensive Connectivity and The Fundamental Theorem of Nonlinear Algebra

The assertion that all hypernumber equations have hypernumber solutions, i.e. that hypernumber-arithmetic is closed, we have dubbed "The Fundamental Theorem of Nonlinear Algebra". The geometric meaning of this conjecture may be visualized as follows. "Solvability" is a question of "arranging transportation" in hypernumber space. It is a question of the "traversibility", "transportability" or "self-connectedness" of that space. Is it possible to get from any point in $\mathbb{H}$ to any other, by means of a path or trajectory induced by an operator which also names a point inside $\mathbb{H}$? That is the question upon which, we believe, the conjectured "Fundamental Theorem" stands or falls. If any two points in $\mathbb{H}$ can be so linked (including self-linkage, linkage of a point to itself via a closed-loop trajectory, as a special case) then we suspect the Theorem holds.

Ideographically, this "Principle of Comprehensive Connectivity" could be written[*]:

$$(\forall b,\ \forall c \ \varepsilon\ \mathbb{H})\ (\exists a \ \varepsilon\ \mathbb{H})\ [a(b) \rightarrow c\,]$$

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[*] (Here ε refers to the set-inclusion relation, not the hypernumber unity epsilon.)

Meaning there is always a "way" (namely, some operation denoted a) within $\mathbb{H}$ to get from any b in $\mathbb{H}$ to any c in $\mathbb{H}$ (including cases wherein b = c). Transportation within $\mathbb{H}$ can always be arranged, using only the resources of $\mathbb{H}$ itself. You can get from anywhere in $\mathbb{H}$ to anywhere (else) in $\mathbb{H}$ via an operation also in $\mathbb{H}$.

9. $\mathbb{R}^n$ and $\mathbb{H}$ as Part and Whole ($\mathbb{R}^n \subset \mathbb{H}$)

In all this discussion, it is well to keep in mind that $\mathbb{R}^n$ is a subspace of $\mathbb{H}$: $\mathbb{R}^n \subset \mathbb{H}$. In contrasting the properties of $\mathbb{R}^n$ and $\mathbb{H}$, we are contrasting part and whole. $\mathbb{H}$, with all subvarieties of (hyper)numbers (i_1, i_2, i_3, ..., ε_1, ε_2, ε_3, ..., etc.) included, is infinite-dimensional. This is so because some of the (hyper)numbers have an infinite number of subvarieties with each subvariety of unity "unit-ing" a distinct dimension of $\mathbb{H}$, a unique axis in $\mathbb{H}$ with a unique directionality, perpendicular to the directionalities of all the rest. In particular, ordinary unity, r, has an infinite number of subvarieties* --r_1, r_2, r_3, ... -- and these are included in $\mathbb{H}$. Therefore Hilbert space, $\mathbb{R}^\infty$, is comprehended within $\mathbb{H}$.

Thus the entirety of Real arithmetic, Real algebra, and Real analysis is "going on" in that special "corner" of $\mathbb{H}$ which is $\mathbb{R}^n$. All of the expressions, equations, and operational processes of that arithmetic, algebra, and analysis form a restricted special case in the midst of the manifold of all possible expressions and processes of the hyperalgebra or process algebra entire.

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*[One way to visibilize the distinctions among subvarieties of r, and the 'meta-dimensional' relationship obtaining between each r-variate axis and its successors in the hierarchical sequence of r subvarieties, is as follows:

$$r_1 \equiv 1; \quad r_2 \equiv \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}; \quad r_3 \equiv \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}; \quad r_4 \equiv \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}, \text{ and so on.]}$$

10. The Space of Logic as a Subspace of $\mathbb{H}$

("HyperBoolean" Algebra and the Analytic Geometry of Logic:
The "Boolean" Core of $\mathbb{H}$ and the Logical Core of Mathematics)

The original Boolean algebra was founded upon the remarkable analogy between the processes of formal deduction -- the processes of Aristotelian logic -- and the behavior of the operators 0 and 1 (0r and "1r" to be exact), which bound the unit interval of the $\mathbb{R}$eal line. Upon encountering the hypernumbers, it is natural to wonder whether the unit intervals of their unities also exhibit "logical" properties. It is natural to ask "Is there a generalization of Boole's logical algebra which utilizes the union of the unit intervals of all ten basic forms of unity, r included, to express a logic?" To the extent that the unit interval of r fulfills the description of Aristotelian processes, it becomes natural to wonder if the other hypernumber unit intervals may be utilized to express trans-Aristotelian processes -- a nonclassical logic.[77]

In the body of this section above, under sub-headings like "Hypernumber Logic", and "The Logic of Differential Equations", we have implied that this is the case: that there is such a generalization and that it embodies a trans-classical, trans-Aristotelian logic which we dubbed "nonlinear logic". We are now in a position to elaborate some of these ideas.

The idea that the unit interval of each kind of number exhibits "logical" properties -- the generalization of the Boolean idea -- has some intriguing implications. The unity or unit-form of each kind of number expresses the operational behavior characteristic of that number-variety in its "purest" form, unadulterated by multiplicity or by additive "admixture" with other varieties of number. "Logic" in this sense means 'Logic of behavior' or 'Logic of operation' -- something akin in meaning to the ancient "logos" concept and to the modern usage wherein we speak of a "dynamic" (or *pattern* of change) as characterizing some entity. Each basic operation ([hyper-]number) *is* or has its own "logic" in this sense, and the unit-form or unity represents that "logic" in its quintessential form. That logic is represented analytic-geometrically by the *shape* of the trajectory characteristic of that number variety: by the power-orbit. But what has this to do with the more familiar notion of logic as rules of inference or *laws of deduction* of one true formula from others? Our task here is to bring these two senses of "logic" together, to their unity.

Viewed most generally, each hypernumber variety represents a distinct *algorithmic pattern* or *pattern of calculation*. The hyperarithmetic can be depicted geometrically. That is, it can be displayed as an organic (differentiated) space. In that space, every number, every possible multiple and admixture of every kind of number, is arrayed, with a definite, defining position relative to all others, forming a *number-manifold*. This number-manifold we denote by "$\mathbb{H}$". When the hyperarithmetic is thus represented geometrically, calculation appears as a movement from number(point) to number(point) in that space; specifically as a movement from the input or *data*-number to the correct answer-number or result. A calculation is depicted as a pattern of movement through that space, a definite trajectory in that number-manifold. Different *ways of moving* in number space represent *different ways of calculating*. The different hypernumber varieties thus represent different basic modes of calculation, different fundamental "algorithms." Let us symbolize a *general* calculation. Let us denote the "input" to or "data" of the calculation by the contents of the parenthesis "()", the "output" or "answer" by *c*, and the "algorithm" or function by *a*. Then, in:

$$a(\cdot) \rightarrow c$$

if our "data," the initial "measurement" input to the algorithm, were $2i_1$, and if our algorithm *a* equals $4\varepsilon_1$, then our outcome would be $c = 8i_0$. However, if our algorithm *a* equals $3i_1$ with the same input, our answer would be $-6r$. Not only would the answer, the endpoint, be different, but the pathway through "calculation-space", the *track* beginning at $2i_1$ and ending at $-6r$, would also differ. In the first case, the track or trajectory would be characteristic of ε numbers in their interaction with i-type numbers. In the second case, the track "left behind" would be characteristic of the interaction of i numbers with their own kind. Of course, we can imagine much more complicated "algorithms" (in general: $A_1r_1 + \ldots + B_1i_1 + \ldots + C_1\varepsilon_1 + \ldots + D_1w_1 + \ldots + E_1p_1 + F_1q_1 + \ldots + G_1\sigma^{+1} + \ldots + G_2\sigma^{+2} + \ldots + H_1\sigma^{-1} + \ldots + H_2\sigma^{-2} + \ldots + I_1m_1 + \ldots + J_1\Omega_1 + \ldots + K_1v_1 + \ldots$) and much more complicated "inputs" (in general: $a_1r_1 + \ldots + b_1i_1 + \ldots + k_1v_1 + \ldots$) also falling under "the general form of calculation" symbolized above. The examples were chosen to keep things simple.

But the key point is this. Looking at hyperarithmetic in this way, geometrically, we see it as embodied in this "space of all calculations" called $\mathbb{H}$. $\mathbb{H}$ is a number-manifold. It is a number*ed*-manifold: a multi-dimensional *expanse* of (hyper)numbered-points. Every point in $\mathbb{H}$ is named by a (hyper)-number. Every number is a possible data-number (input-number), or answer-number, or algorithm/operator in any number of possible calculations, or arithmetic processes -- representable by hyperarithmetic expressions of the form "a(b)" [=c] -- that can go on in $\mathbb{H}$.

Thus $\mathbb{H}$ is not just a space of ("containing") *numbers*. It is a space "containing" all possible *connections between* numbers as "datum" and "answer"; as input and output. $\mathbb{H}$ is a *space of algorithms*, a *space of* "*connections*". The algorithms or connections are represented geometrically by the trajectories connecting the input and output points (numbers); joining the data to the answer points. To be precise, it is the abstracted "silhouette", the general

shape

shape of the trajectory which expresses the essential character of the algorithms displayed as a space, $\mathbb{H}$. This summary exists abstracted from any *particular* interpretation or meaning. It has *many potential* applications or interpretations. It can be used to express many possible particular meanings. For example, $\mathbb{H}$ can be interpreted as a state-space -- the application we have emphasized here. The "connections" selected can be interpreted as the state-space *trajectory* of a particular dynamical system, the algorithm formulating those "connections" as the "state-function" of the system. The right "algorithm" for a given application is of course the one that consistently connects the given data to the *correct* answers; the one that has the right *calculative-trajectory* for the problem in hand. $\mathbb{H}$ and the hyperarithmetic simply provide an ideographic-linguistic facility for expressing such interconnections, a versatile "trajectory language."

Each number-symbol of the hyperarithmetic, each "point" in $\mathbb{H}$ has the symbolic form "nu_k", where n denotes a cardinal number and u_k an elemental hypernumber unity, or is of the form of a "sum" of such forms. That is, each number/point has the symbolic form of a *multiplicity* (n) qualified by a *trajectorial predisposition indicator* (u_k). The latter indicates the trajectory potential of that point, the type of movement "congenitally" arising from that point should it be taken as a starting-point (datum, operand, argument, input) of an algorithm (operator, function) (this movement of course also depending upon the way the u_k or u_k's of *that* algorithm characteristically interact(s) with the u_k('s) of the starting-point.)

If we take seriously the idea that the unit-hypernumbers -- that is, the trajectory-qualifiers or indicator-tags "u_k" -- express new dimensions of *logic*, the observations foregoing suggest a new conception of the connection between logic and "mathematics proper." They suggest that logic is the essential *core* of mathematics [-- at least of all the mathematics which can be represented in ℍ , which includes ordinary arithmetic and algebra, the linear algebra of matrices, plus the expanded "algebra" of integrodifferential equations.] By "logic" here, again, we mean "Hyper-Boolean" algebra: the arithmetic and algebra of the unit-intervals of the ten basic hypernumber unit operators -- r, i, ε, w, p, q, Ω, m, σr, v -- *interpreted* à la Boole as an arithmetic and algebra *of logic* (in a manner still to be described).

It is not a question of reintroducing a Russellian logicism. We are not suggesting that all of mathematics can be derived from formal logic alone via the rules of inference or deductive processes *of that logic itself*.* We mean first of all that the operational, behavioral "logic" of the pure unit operators is at the "core" of all the calculational phenomena of the multiple, compound, and "dynamical" or temporalized forms of ℍ arithmetic and algebra in general. As noted previously, all quantities occurring in ℍ-arithmetic are "tagged" with a hypernumber "qualifier," an indicator of *operational* quality. Every number is thus labeled, suffixed according to its power-orbit type, according to the type of self-trajectory or "flow-field" to which it belongs.** Thus arithmetic expressions can be typed according to the hypernumber unit qualifiers,

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*[Dr. Musès has remarked that the hypernumbers could never have been effectively *deduced*, by varying the "eight basic axioms of ordinary (r) numbers" in various permutations and combinations, and determining by logical means which variants yielded viable (i.e. self-consistent) arithmetics and algebras. Each hypernumber variety abrogates or deviates from these eight laws of Real numbers in a complex and unique manner. Since the possibilities of deviation and *partial* abrogation are manifold for each law, trying to deduce or axiomatically determine the viable patterns of abrogation would involve a huge, if not a "transcomputational," search tree. According to Dr. Musès, geometric (power-orbit) considerations were crucial in narrowing the search.]

**[This lends further coherence to the idea of the mathematical point as a relative, not an absolute, independent, isolatable entity; as a moment of a flow, whose meaning is determined only by its location *relative to* the entire manifold of (other) operator-points. See p. 93 above and observation 11 below.]

the operational qualities, that occur in them, regardless of differing multiplicities. Arithmetic formulae involving strikingly different multiplicities and dynamicities can nonetheless belong to a single "logical" ("Hyper-Boolean") type. For example

$$\text{at } t=1, \quad (3i_1 + 2i_2 + 4\varepsilon_3)^t \cdot (i_1 + \varepsilon_1) = (-3r_2 + 3i_0 - 2i_3 + 2i_1 + 4\varepsilon_2 - 4i_2)$$

whereas

$$\text{at } t=0, \quad (60i_1 + 24i_2 - 14\varepsilon_3)^{2^t} \cdot (1.5i_1 - \sqrt{20}\,\varepsilon_1) = (-90r_2 + \sqrt{7200}\,i_0 - 36i_3 + \sqrt{11520}\,i_1 - 21\varepsilon_2 - \sqrt{3920}\,i_2)$$

Yet both of the above belong to the same Hyper-Boolean "logical" or operational category.*

This is another way of saying that "logic" abstracts from the concrete, determinate details of multiplicity and dynamicity (temporality), seeing only the root operations involved. In both of the above, logic "sees" the same thing.

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*[In matrix form, the two expressions above become:

$$\left(3\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} + 2\begin{bmatrix}i & 0\\0 & -i\end{bmatrix} + 4\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right)^1 \cdot \left(\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} + \begin{bmatrix}0 & i\\-i & 0\end{bmatrix}\right)$$

$$\text{or } \left(\begin{bmatrix}0 & -3\\3 & 0\end{bmatrix} + \begin{bmatrix}2i & 0\\0 & -2i\end{bmatrix} + \begin{bmatrix}0 & 4\\4 & 0\end{bmatrix}\right) \cdot \left(\begin{bmatrix}0 & -1+i\\+1-i & 0\end{bmatrix}\right) = \begin{bmatrix}2i & +1\\+7 & -2i\end{bmatrix} \cdot \begin{bmatrix}0 & -1+i\\+1-i & 0\end{bmatrix},$$

$$\text{and } \left(\begin{bmatrix}0 & -60\\60 & 0\end{bmatrix} + \begin{bmatrix}24i & 0\\0 & -24i\end{bmatrix} + \begin{bmatrix}0 & -14\\-14 & 0\end{bmatrix}\right)^1 \cdot \left(\begin{bmatrix}0 & -1.5-\sqrt{20}i\\+1.5+\sqrt{20}i & 0\end{bmatrix}\right) =$$

$$\begin{bmatrix}24i & -74\\+46 & -24i\end{bmatrix} \cdot \begin{bmatrix}0 & -3/2 - i\sqrt{20}\\+3/2 + i\sqrt{20} & 0\end{bmatrix}.]$$

It sees only i_1, i_2 and ε_3 operating upon i_1 and ε_1. Indeed, the operational logos of i_1, of i_2, of ε_3 and ε_1 form the common "skeleton" of both expressions, "fleshed-out" differently by different multiplicity "modifiers" (coefficients) in the two cases. Indeed, the compounded unit operators $i_1 + i_2 + \varepsilon_3$, and $i_1 + \varepsilon_1$ form the arithmetic "core," the operational essence, of both the above calculation-formulas, despite their more superficial, "cardinal" differences.

The realm of "Hyper Boolean logic," represented ideographically, is:

$$[0,r] \cup [0,i] \cup [0,\varepsilon] \cup [0,w] \cup [0,p] \cup [0,q] \cup [0,\sigma r] \cup [0,m] \cup [0,\Omega] \cup [0,v] ,$$

the union of all the hypernumber unit interval sets with the original Boolean r unit-interval set. Looked at geometrically, not only is this a subspace of $\mathbb{H}$, it is literally the "core"* of $\mathbb{H}$:

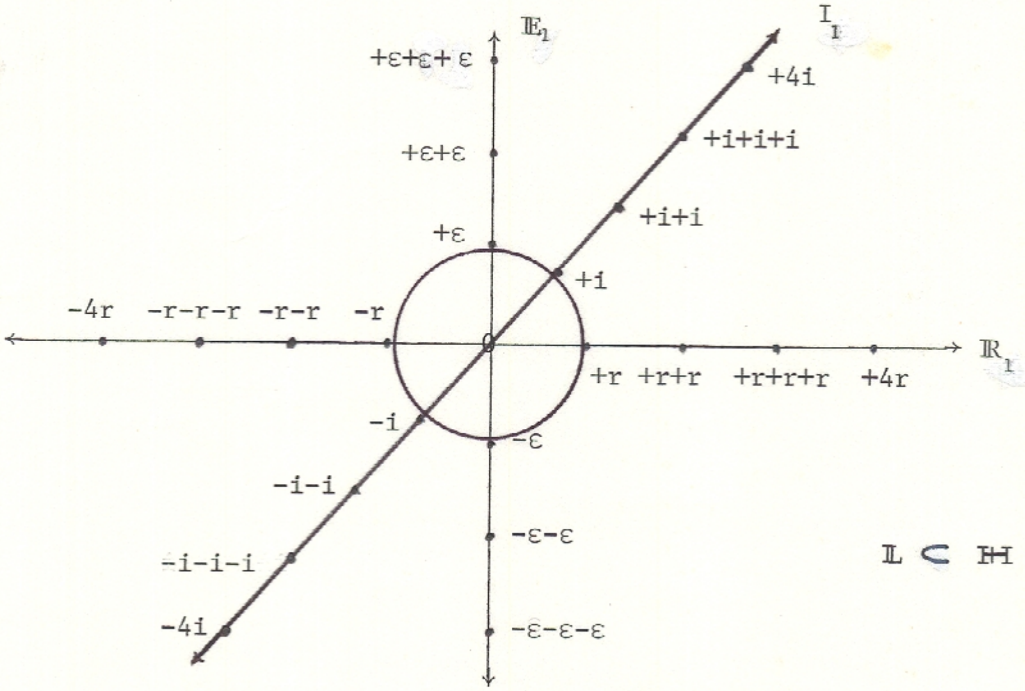

Figure 10: $\mathbb{L}$, The "Logical Core" of $\mathbb{H}$ (3-d excerpt only)

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[*] [We are including the negative unit intervals in the "sphere" of logic, whereas Boole explicitly excluded them[78], though he did admit "logical subtraction" in his algebra of logic. The possible logical meaning of negative values will become clear below.]

Here, of course, since we are able to depict only three axes of $\mathbb{H}$ at once in any one drawing, we have selected the r_1, i_1 and ε_1 axes (the 1-d spaces $\mathbb{R}^1_1, I^1_1$ and $\mathbb{E}^1_1$) as the exemplary excerpt from $\mathbb{H}$. The region enclosed by the (hyper-) sphere inscribed at the center, around the origin of the space, is our hyperBoolean "space of generalized logic," the unit-radius *core* region of $\mathbb{H}$ itself.

But the question remains: in what way does this "hyperBoolean logical arithmetic" relate to, for instance, first order and higher order predicate calculus; that is, to logic understood as a formal language for expressing propositions and the inferential, implicative forces connecting them? How could hyperBoolean algebra be used to formulate a chain of deductions, a formal proof? How can $\mathbb{L}$-arithmetic be interpreted as a calculus of propositions?

One answer, consistent with the general "dynamical systems" approach to the utilization of $\mathbb{H}$ which we have pursued throughout this section, runs as follows. The key idea is to grasp a proposition, a *formula* of a suitable version of symbolic logic, as representing a state of mind or, more precisely, a state of *cognition*. That formula must then be associated with a unique point in $\mathbb{L}$. A formal proof is written down as a sequence of formulas, a "chain" of propositions. The proof is thus represented in hyperBoolean by a sequence of points in $\mathbb{L}$, the successive points corresponding to the successive formulas. The proof is pictured as a trajectory. The trajectory depicts the "line of reasoning," the "course of argument," the "path of inference," the "pattern of derivation" employed in the proof. It *literalizes* these traditional "figures of speech" about logical processes. This approach treats the formal proof, the deduction-process as a dynamical system. It treats logical processes as dynamical, temporal or time-consuming events; as actual "material" processes of mind or of cognition (whether or not the exact timing of the trajectory is made explicit by the superscript-function procedure).

It should be noted that the "trajectories of thought" thus depicted are *not continuous* trajectories. Proofs usually "skip steps." Many additional formulas could be introduced as intermediate stages of inference between any two that actually appear in a typical proof. However, seldom could such "interpolation" be stretched to supply even aleph-null intermediate formulas, let alone a continuum-number of them. The trajectories will thus be discretized, quantized. The movement of thought thus modeled appears to "jump" from point to point, just as the mind, the attention, skips from formula to formula, proposition to proposition, meaning to meaning in perusing the written proof.

But how do we assign a proposition to a (unique) point in $\mathbb{L}$? How do we associate any possible "state-ment" of symbolic logic to a definite state-point in this core subspace of $\mathbb{H}$, thus interpreted as a "logical state-space" or "state-space for logic"?

The method we propose is a conjecture, and remains to be worked out in detail. It involves the whole question of formula-description and formulo-"metry," of the description of the ideographic formulas of symbolic logic and of the "measurement" of their syntactic and semantic properties. It might best be presented as a generalization of the "Gödel-numbering" scheme used in establishing the famous "Incompleteness Theorem."[79] Gödel-numbering assigns a unique natural number (positive integer) to every possible syntactically-correct formula of a given version of symbolic logic. Such a formula consists of a definite number of standard symbols -- parentheses, connectives, letters for variables, etc., arranged in a certain sequence:

$$(P^0 \vee P^0) \rightarrow P^0$$

Call this number m (in the example above, m = 10). Each standard symbol is assigned a definite cardinal number. The Gödel number of a formula is found by associating the first m prime numbers, in ascending order, from left to right, with the standard symbols forming the formula, raising each prime to a power which is the pre-assigned cardinal number of its associated standard symbol, and taking the product of all these powers of primes. The power of each prime is determined by the number assigned to its corresponding standard symbol. Which prime gets that power depends on the sequential order in which the standard symbol occurs in the formula, and the symbolic "length" (m) of the "standard character (symbol) string" composing the formula. That is, Gödel-numbering responds to the _syntactic_ features of a symbolic-logical expression and thus, in a sense, gives a "measurement" of the overall syntactic quality or typology of that expression.

But Gödel-numbering measures all formulas along a single axis or dimension, the r-axis (with only positive integer values possible as formula-measures). We want to distinguish several different dimensions of formula-description. Each would concentrate on "measuring" a special aspect of logical formulas. In particular, _semantic_ as well as syntactic features would be measured. All formula-measurement values would be "normalized" to fall within the unit-interval. That is, the maximum measurement-value would be 1 unit, the minimum zero.* These different measures of "logical state" represented by a formula, would be the logical "state-variables" of $\mathbb{L}$, with $\mathbb{L}$ interpreted as the "state-space of logic," or "truth-space." We say "truth-space," or "truth-state-space" because every formula occurring in or "asserted" as a step in a proof must be a true formula, a "truth-state" of the logical symbolism. The logical "state-space" proper is that sub-space of $\mathbb{L}$ whose points represent _true_ formulas. Only true formulas are _possible_ formulas -- possible occurrences in the proof-chain. Therefore only they are possible _states_ of that "dynamical system" constituted by the proof. And, by definition, the "state-space" proper is the set of all possible states of the dynamical system being modeled by ("in") it. It is the space consisting of all points representing accessible, meaningful states of the system.

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*[This would include fractional and possibly irrational measurement-values on the unit-interval.]

Which hypernumber axes these various formula-measurements will be assigned, which hypernumber unity will serve as the "qualifier" of what state-variable, as the qualitative "unit of measure" in which the measurement-numbers quantifying a given quality of logical formulas will be counted, will vary. It will depend on the specific "dynamics" of the argument being represented, on the particular "logos" of the proof being thus modeled. That is, it will depend on the type of movement, the shape of trajectory,* which the particular proof-argument expresses, given the formula-measures, or "descriptions" of formula-states, being employed.

In sum, the method consists in depicting the logical process of "deduction" or "derivation" as motion in logical (state-)space. Once again, we have a "nonlinear strategy of symbolization." The logical process is first, directly represented by a sequence of ideographic logical formulas, a proof. Each formula represents a node of the thought process as attention retraces its way through the argument. Each formula-node is then translated into coordinates of a point in $\mathbb{L}$ using the "hyper-Gödelian numbering" formula-measures discussed above. The sequence of logical formulas is thus translated into a sequence of "formula-points" representing successive states of cognition, i.e. states of the mental logic-process, encountered in thinking through the proof. Diagrammatically:*

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*[Since the logic-trajectories are discontinuous, there must be specified a way to, so to speak, "fill-in the gaps," between formula-points along a trajectory so as to ascertain its implied shape, and thereby determine the appropriate hypernumber involvement in its formulaic expression. Such formulaic expression also requires special symbolic machinery, effecting discontinuities in the superscript-functions of the usual reflexive and flexive hyperfunction form, which otherwise describe continuous paths. Similar problems of discontinuity arise in constructing the geometry corresponding to the original Boolean logical arithmetic and algebra, for the purpose of graphing Boolean functions expressing propositions. This "analytic geometry" of logic gives discontinuous graphs because only the endpoints of the $\mathbb{R}$ unit interval, 0 and 1, and not any of the interior points of that interval, are allowed as explicit logical values in Boole's system. Of course, the implied continuity or continuous completion of such discontinuous Boolean graphs are exclusively rectilinear; either vertical, horizontal, or 45° diagonal straight lines.]

$$\begin{tikzcd} \text{TRAINS OF REASONING} & \text{CHAINS OF LOGICAL FORMULAS} \arrow[l, bend right] & \text{TRAJECTORIES IN LOGICAL SPACE} \arrow[l, bend right] \end{tikzcd}$$

In line with Boole's original programme, what we are modeling, through the intermediaries of logical formulas and $\mathbb{L}$-trajectories, is the movement of thought itself. The trajectories depict mental operations. The hypernumber operators become "logical numbers," logical operators. Interpreted in this way, the unit operators stand for mental operations. They form a thought-process algebra, an algebra of ideations, of logical, conceptual events, a calculus of cognition. Their arithmetic models the dynamical system of the mind. First, translating ideas and idea-movements into symbolic-logic formulas and formula-sequences makes the ideational process measurable, quantifiable. Then, translating these measurements into trajectories, thus translating the dance of ideas into a dance in $\mathbb{L}$, indicates the formulation of this dance in terms of the (hyper-)numerical expressions of hyper Boolean "logical arithmetic."

In this light, we can understand the meaning of "nonlinear logic." Provided the formula-measurements are well-chosen, formal deduction-processes should map to certain definite types of trajectories. It may be found that there exist other types of logical trajectories which represent quite valid and useful "paths of inference," but which do *not* correspond to the types of *inferential trajectory* singled out by classical logic. Such "nondeductive, non-Aristotelian" trajectories might correspond, for example, to "creative processes," i.e. processes leading to the first-time synthesis of *new* concepts, new hypotheses, new arguments and theories, or to the achievement of new discoveries, as well as to "intuitive leaps of insight" and processes associated with meditative states, etc.

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*[As before, the arrow is directed from the thing representing to the thing represented. cf. pp. 80-81, above.]

In view of the manifold associations of classical Aristotelian logic with the "logos" or logic-of-operation of r, the "rectilinear operator",[*] and with linear differential equations, both in the original Boolean algebra and elsewhere,[**] we might suspect that these "non-Aristotelian trajectories" belong to the hypernumbers, with their non-rectilinear flows, and especially to the nonlinear (nondistributive, noncommutative) numbers[†] among them. In that case, the $\mathbb{L}$ space would be able to model "logic" in a most expanded sense. It could model *the conceptual process* of mind with a scope not restricted to those processes of deductive reasoning so far codified in traditional symbolic logic. It could model any train of thought or "stream of consciousness" which a sufficiently rich version of symbolic logic -- perhaps including modal and temporal operators, and more[80] -- was capable of expressing via a definite sequence of formulas.

The basic *operations* which the hypernumber unities symbolize are represented, geometrically, as various choreographies or kinds of movement in $\mathbb{L} \subseteq \mathbb{H}$.

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[*] [r-number flexive functions $a^{f(t)}(b)$, $a \in \mathbb{R}^1$, induce *rectilinear flows* in $\mathbb{H}$, movements confined to a single dimension of $\mathbb{H}$; namely, to that dimension or axis, radial to the origin, to which *b* belongs.]

[**] [See pp. 59-61 above.]

[†] [See pp. 63-65 above.]

The interpretation of $\mathbb{L}$ as a "dynamical analytic geometry of logic" uses those movements to represent possible pathways of thought; elementary patterns of inference; elements of conceptual processes, operations of mind. Corresponding to this special interpretation of the geometry of the hypernumber unities, is a special interpretation of their arithmetic and algebra. All hyperarithmetic computations and calculative processes are interpreted as "deductions" or inferences (not necessarily of the "Aristotelian" kind). Juxtaposition now connotes derivation. That is, the general process-form $a(b) \rightarrow c$ is interpreted as a logical process;

$$a(b) \rightarrow c$$

meaning: "$\underline{a}$ derives $\underline{c}$ from $\underline{b}$"

> or, more precisely, "the formula rated $\underline{a}$ (the formula corresponding to the point $\underline{a} \in \mathbb{L}$, i.e. with coordinates $\underline{a}$), acting upon the formula corresponding to point $\underline{b} \in \mathbb{L}$, yields the formula corresponding to $\underline{c} \in \mathbb{L}$."

This assumes that a formula(-point) embodies a "summary" of previous derivation-movements arriving at it, and hence, already, inherently expresses a tendency or "potential" regarding further derivation, a preferred pattern of inference, as if a luminiferous field of implicational "lines of force" surrounded it. Thus, "applying" one formula to another formula amounts to applying a definite strategy or "direction" (flow) of inference to that other formula, thus removing to another formula as conclusion or result, i.e. to another point in $\mathbb{L}$, named (measured) by another hypernumber, c. The implicative "force fields" of the two formulas interact, moving the thought-system to a new state. The type of logical movement engendered will, of course, correspond to the type of operational movement characteristic of the hypernumber $\underline{a}$. Thus, in this model, formulas interact with formulas to yield new formulas. For example, listing two formulas consecutively in a "proof" or argument suggests a third, which is written down next:

		FORMULA	JUSTIFICATION	HYPERNUMBER MEASURE (Hypothetical)
	(1)	E^2xy or* $x = y$	given	a
	(2)	E^2yz or $y = z$	given	b
conclusion:	(3)	E^2xz or $x = z$ Q.E.D.	transitivity	c

or:

$$(x=y) \& (y=z) \rightarrow (x=z)$$

[Here the logical "and" sign "&" functions as "logical multiplication" or like our juxtaposition:

$$(x=y)(y=z) \rightarrow (x=z).]$$

Given our interpretation of formulas, this posits concepts as "mental operators" or "mental fields": "ideas interact with ideas yielding (new) (other) ideas."

The approach interpreting calculation as logical derivation, even at the level of (trans-unit-interval) mathematics proper, is echoed in the following quote from a recent ground-breaking work on economic modeling. In addition, this passage raises simultaneously and in that "operational-ideographic" context, the notion of "circular logic" and its usefulness:

> A circular definition, a definition that defines some concept in terms of itself, *idem per idem*, was considered a grave fault by classical logic. But self-contained (closed) systems require circular definitions and modern scientific experience has demonstrated their power.
>
> Classical labor theory [of Smith, Ricardo, et. al.] thought itself free of such "logical blunders." It defined values by reducing them to labor expenditure, explaining them as a certain amount of crystallized labor energy. The corresponding mathematical equation was $\underline{p}(1-A) = \underline{v}$, which could be solved as $\underline{p} = \underline{v}(1-A)^{-1} = \underline{v}Q$. That is, it started from [the vector of] direct labor expenditure(s), $\underline{v}$, the *premises*, and from these *premises*, using

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*["E^2" stands for the binary relation of "equality."]

> the operator* (matrix) Q, it deduced the conclusion, p [p is the vector or "list" of prices (here = values) of all commodities of the economic system]. The prescription was to add up all the labor directly and indirectly expended on the product. This logical operation yielded the sought values.
>
> The eigenequation* for the closed model, $\underline{p} = \underline{p}A$, leading to the same numerical results, would have been considered circular by classical logic. It starts from p and deduces its own premises, p, again. It defines values by values, idem per idem, moving in a circle rather than reducing values to something given previously. Nevertheless it is still a logically correct and fruitful formulation and in a sense a more general one than that for the open classical model.
>
> Böhm-Bawerk [1896] was the first of a long series of critics who objected to this sort of reasoning. Rebuttals were unconvincing because they attempted to deny circularity altogether... A more realistic and fruitful defense would acknowledge the circularity but go on to demonstrate the merits of such definitions in scientific economic thought.... Mathematical difficulties arise when we try to eliminate circular definitions.[81]

We will consider the "circular" price-system model $\underline{p} = \underline{p}A$ and its dual production-system model $A\underline{x} = \underline{x}$ as models of economic reproduction in greater detail below, in part 4. What concerns us here is the issue of circular logics. As noted previously (see pp. 38-9 below), the simply circular paradigm is insufficient to model an economy characterized by expanding (or even by contracting) economic reproduction, or to model any evolving, self-developing system. Indeed, the equations discussed in the passage quoted above refer to the extreme idealization of "simple reproduction," The equations $\underline{p} = \underline{p}A$ and $\underline{x} = A\underline{x}$ are linear equations (matrices like A are linear operators), and they model a state of economic equilibrium in which p and x represent stable "fixed points" in the spaces of values and products respectively. Such stable fixed point solutions are extremely rare among reflexively nonlinear dynamical systems, for which limit-cycles, strange attractors, and branching-solution "jump phenomena" predominate. To model expanding reproduction, even if only in terms of merely *quantitative* growth, these equations would have to become $\underline{x} > A\underline{x}$ or $\underline{x} - A\underline{x} > 0$ (meaning the total social product output, $\underline{x}$, is greater than the input consumed to produce that output, $A\underline{x}$) and $\underline{p} > \underline{p}A$ or $\underline{p} - \underline{p}A > 0$ (meaning the value of the total output of the economy, $\underline{p}$, is greater than the value of the inputs consumed in producing it, thus leaving a *surplus* value available for reinvestment to expand production).[82] In both cases, therefore, the circularity is thereby lost. To express phenomena such as crisis and *qualitative* growth -- the emergence of new systemic qualities in the continuum of system evolution -- would require something more than comparative statics (statical system theory) and linear equations involving exclusively linear operators can encompass.

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*[The formulas here deviate from our norm in that the "operators" (matrices) Q and A respectively, are juxtaposed to the right instead of to the left of the "operands," the vectors v and p respectively. But this makes sense in terms of the conventions for matrix-vector multiplication. Matrices can have both "right" and "left" eigenvectors, and here p is a left eigenvector of A, for the eigenvalue 1.]

We need, at minimum, the paradigm of "*helical* circularity" to encompass the phenomena of *expanding* economic reproduction and *crises* of qualitative growth. What would we require of a *logical* system to deem it capable of metaphorizing or modeling such phenomena, or even of qualifying as a "logic of development"? We would still want, I believe, a "self-confirming" or "self-reinforcing" system of premises, that is, a system of premises setting up a "curved" logical universe, such that all paths of inference from the premises would wind up back at the premises again. But this return would have to have a *cumulative* dimension. We would want such a logical system to be not only self-confirming but *self-amplifying* as well. That is, the "self-deduction" of the premises would have to be accompanied by a kind of "logical *gain*." As a result of the current of deduction turning through the helical armature of this "curved" universe of discourse and returning upon its source-premises, the premises would have to be enriched, perfected.[*] They would have to become more true, more revealed, as a result of that excursion. Their self-re-derivation ultimately such a logical system should come to a "singularity," a point where the "potential logical energy," the "truth-potential" of the premise-system would be fully actualized, exhausted. At that point a new system of premises with new rules, a new "geometry" of inference incommensurable with that of the old, might arise as the final product of the old.

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[*] [Applications include Robotics. Various values in axioms and theorems of a Robot's behavioral premise-system would be assigned by *sensor input*; would be *variables* in the core operator P or P(t). The behavior-system can then be improved not only for endless particular cases, but at the highest level,

Suppose we denote by P the system of premises of such a developmental logic-model. Suppose further that we conceive P in "Nootkan" fashion, as denoting simultaneously a system of formulas from which derivation may be made, and a tendency of derivation, a derivational predisposition; i.e., a derivation operator enforcing a certain method and direction or "mode" of inferential movement. If we again use juxtaposition to indicate activation of a logical operator or "deductor" upon a logical operand (a premise-formula, or system of premise-formulas), then we recover our standard reflexive function form as the ideographic, "algebraic" description of such a helically-circular logic-system:

$$P(P) \rightarrow P^2;\ P^2(P^2) \rightarrow P^4\ ;\ P^4(P^4) \rightarrow P^8$$

$$\ldots\ |P| < |P^2| < |P^4| < |P^8| < \ldots$$

or, succinctly:

$$S_L(\tau) = |P^{2^\tau}|$$

$$\text{or}\quad S_L(t) = |P^{f(g(t))}| = |P^{fgt}|$$

Here τ is a discontinuous or quantized "step function" counter that varies with the successive stages or "generations" of self-deduction of the premises. $S_L(\tau)$ stands for a measure of the state (of self-reflection or development) of the logical premise-system at stage τ. The "g(t)" measures observed values of the "logical gain," which the function f translates back into the appropriate degree-of-reflection, in the case of "real-time" modelling applications.

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that of the premises themselves. These can be continually self-evaluated on the performance flowing from them as a whole in reference to the unit's 'experience'

Summary on the Logical Application of $\mathbb{H}$

$\mathbb{H}$ provides the arena in which the computational processes of ordinary arithmetic, algebra, and functional analysis as well as hypernumber arithmetic, including matrix and vector "arithmetic" and algebra (linear algebra), can be depicted geometrically. We regard integrodifferential equations as an extension of ordinary algebra, to include (i) the integrodifferential operators σ^n ($n \in Z$) explicitly; plus (ii) the other hypernumber operators implicitly (in the form of hyperfunction solutions), thus including "nonlinear algebra"; and (iii) "temporalized" or "dynamical arithmetic," that is, functions of time. Thus $\mathbb{H}$ is the space where most of mathematics is happening.

We are proposing that $\mathbb{L}$, a subspace of $\mathbb{H}$, be interpreted for logic, in the sense of a dynamical, and in other ways generalized version of Boolean algebra. $\mathbb{L}$ is at the literal core of $\mathbb{H}$. It is the heart region of $\mathbb{H}$, enclosing the origin. Thus, geometrically speaking, this logic seems to reside at the core of mathematics. And this logic is no less central to the formulaic side of mathematics: hyperarithmetic and hyperalgebra. The hypernumber unit qualifiers attached to every distinct multiplicity occurring in hyperarithmetic expressions form the operational "marrow" of every calculational process as well. Extracting or abstracting these unities from a given expression, leaving the detailed data of multiplicity behind, still gives a good assay of the essential computational meaning and behavior of the expression. On both counts, then, logic appears as the core of mathematics.

This is not, however, a recrudescence of Russellian logicism. The relationship between logic and mathematics proper implied here is not one of deductive derivation but of immanence or inherence. Logic is always present in every calculative process as its inner substratum and behavioral essence.

[(cont'd. from p. 147): as a whole; evaluated with respect to the premises themselves, to their own embedded goals, predictions, and expectations.]

And here by "logic" is meant something much wider than Russell intended, something closer to "logos" or "behavioral principle" -- signifying "the dynamics of *ideas*" as conveyed via operator ideograms, in a way which encompasses far more than the classical deductive dynamic.

Traditionally, logic abstracts from the details of multiplicity. Logical quantification is restricted to three categories: some, none, or all. Traditional logic also abstracts from temporality. The arithmetic of $\mathbb{L}$ is the unit-operator core of all $\mathbb{H}$ arithmetic. It embodies the Boolean principle restricting logical arithmetic to unit-interval arithmetic, but generalizes that principle to include all the hypernumber unit intervals in addition to $[0,r]$. As a model of logic it thus fulfills the abstraction-from-multiplicity criterion. But we have called it a *dynamical* version of Boolean algebra. $\mathbb{L}$ is used as a *state-space* for logic, and a logical argument is treated as a *dynamical system*, a *trajectory* in $\mathbb{L}$. Thus, $\mathbb{L}$ logic is a dynamical, *temporalized* logic. Temporality is a part of this logic. The hypernumber reflexive or flexive function describing a trajectory-of-argument sums up the sequence of logical formulas representing that argument in a *single* formula, by a variant of the standard superscript-function procedure. Thus, temporality is implicit in the $\mathbb{L}$ formulation of chains of reasoning. At minimum, time subtley enters via the summary specification of the *sequence* of propositional formulas by the hyperfunction trajectory-description. At maximum, this hyperfunction might even model the "real-time" timing of a mental trajectory, a process of thought. Thus $\mathbb{L}$ logic is temporalized logic, logic which takes time into account, just as the actual thought-process, the actual dynamics of ideas, takes time. In this sense, the $\mathbb{L}$ model fulfills the original Boolean programme in a stronger form than Boole himself envisioned. It provides not only a representation of "mental operations" as a calculus, but a calculus whose calculations could potentially go so far as to mimic or simulate thought-processes -- including "*in*formal" aspects -- in their concrete "real-time" temporality. In addition, $\mathbb{L}$ provides facilities for formulating trans-Aristotelian dimensions of logic, including "circular" and "helical" aspects of logic, via the trajectory-language of the hypernumber metadimensions included in $\mathbb{L}$.

11. *How To Visualize $\mathbb{H}$*

Given the anisotropy of $\mathbb{H}$, the *intrinsic* fields and flows which characterize -- should we not say *constitute* -- its various "wings" and planular "organs," the subplanes and other subspaces of the various power-orbits; given the crucial significance of the *relative* position of each operator-point in the total expanse of $\mathbb{H}$-points, the concept of $\mathbb{H}$ as a point-set, a mere collection of pre-existing or independently subsisting points, falls short of adequacy.

In quantum mechanics, physical space is visualized as continually simmering with "virtual particles," particle-antiparticle pairs momentarily emerging from the "Dirac Sea" of the "vacuum," only to annihilate and return again into the vacuum state. Likewise, $\mathbb{H}$ can be conceived as if a fluid forever flowing, a seething space ceaselessly streaming with "virtual flows"; a space *composed* of virtual "processes," virtual trajectories. $\mathbb{H}$ can be likened to a great sea, unresting, ever stirring and glistening with many tides and crosscurrents in all directions; or to a new river being born, a flood-front with many swirls and rivulets, endlessly pouring forward over fresh ground along the direction of the ever-lengthening t-axis.

The hypernumbers should be imagined not as fixed points, but as "constantly rotating through their power-orbits."[82.5] The points of $\mathbb{H}$, as *still* points, are only the *end*points of the *movements* or trajectory-elements which represent the hypernumbers as *operations*. These endpoints merely take the symbols for the movements which end at them as their adopted names. Only a kind of stroboscopic effect justifies the image of $\mathbb{H}$ as a static expanse of points.

In this continually flowing space, every possible trajectory is always occurring, always available to be singled out as the focus of exclusive attention and exclusive visibility, to use as a metaphor or model for whatever particular process of system-evolution forms the concern of the moment. Thus $\mathbb{H}$ may be visualized as a literal "space of processes."

A Set-Theoretical Metaphor for Our Modeling Paradigm

The "set" concept takes for granted the existence of radically separable, independently constituted and isolatedly subsistent entities, called "logical individuals," "elements," or "objects." It assumes as given precisely what, from the point of view developing here, most requires explanation: the self-differentiation of the existential field, or experienced reality, at every level, into a multitude of relatively and temporarily distinguishable formations.

The set conception is very "nounistic" in the sense elaborated previously, paying little heed to the "verbistic," processal, "eventitial," dynamical aspects of objects. The set concept tends to correspond to an atomistic ontology, rather than a "holistic" or "fieldistic" one.

The need for a transatomistic, "fieldistic" approach for our modeling is most evident in the phenomenon of economic value. The obsolescence devaluation effect, central to our hypothesis, is a striking manifestation of the non-atomistic character of value. Economic goods are not independent, isolatable "particles" of value. Their value is not a localized "substance" inhering in their physical material. Their value is utterly context-dependent. The value of a commodity expresses the relation of that commodity to the totality of commodities, itself included. Thus it has been held that value is an impredicative[83] relation, therefore a reflexive or self-referential entity, and paradoxical in the sense of Russell and Poincaré.[84]

In the above formulation, perhaps a better word than "relation" would be "interaction." And "the totality" (of commodities) should be grasped as a dynamic whole, as the movement of production, consumption, reinvestment, and technological development; as the whole process of "the production of commodities by means of commodities."

Value expresses the dynamic connection of a particular product to the whole, evolving system of production and changing productivities -- to the entire manifold of the economy grasped as a dynamical, nonequilibrium "eventity." That is, value measures the relationship of a particular economic object to the total process of the economic reproduction of society, to the ongoing event of social existence. To mentally "collect" all commodities of an economy with values "affixed" would fail to give an accurate picture of the economic situation, if each value were regarded as belonging to each commodity in itself, separately. A better approach would picture a web-work, connecting every commodity with (virtually) every other; a network of "lines of value-altering force." This "field" of connections would represent the interactive nature of value, weaving the separate commodity "nodes" into a single fabric of inter-determination of value.

Consider, for example, an integrated circuit. This thin wafer of refined germanium, silicon, plastic, and metal would have little if any value in a society lacking a congruent infrastructural context. Unless this object and its production-process coexists, in parallel, with other production-processes producing the material context appropriate to its utilization -- including metal chasis, copper wire, CRT tubes, electricity and much besides -- the integrated circuit does not represent economic value. The object does not represent a definite value in itself, in abstraction from any context, nor in a context to which it is not appropriate.

However, in connection to an economic process providing the above-cited qualities of congruent facilities in sufficient quantity, this tiny wafer might be the CPU of a powerful microcomputer, a "dense" piece of capital worth hundreds of dollars. Somewhat later, in the context of a societal technology and infrastructure transcending semi-conducter methods, this "same" object might resume its former economic nullity. Such an infrastructure is, in fact, already in the germinal stage. In it, "photonics" would supplant electronics. "Light pipes" or "optical fibres" streaming with lased (coherized) photons would supplant metal cables streaming with electrons.

Economic value is a dynamic relation, an interaction. Actual values fluctuate continually, in resonance with the ever-changing, but coherently developing necessities* posited by the evolving life-process of a unique, self-developing species, as well as with the progressively rising productivity of social reproductive activity.

Given these misgivings about set-theoretical concepts, it may seem surprising to find us presenting here a conceptual object which amounts to a set-theoretical model of our basic process-paradigm, our paradigm of self-development. Perhaps more surprising still is the centrality of this object in the theory of sets. But once again, it is important to observe how the dynamic, the reflexive, and the holistic emerge immanently, even where they appear to be, by presupposition, explicitly excluded. Indeed, utilizing only the most natural definitions, we will find that this object is described by a reflexive function, one of the same form we got previously by means of "process algebra" and "operatorial ideogrammar" considerations alone (see pp. 39 and 75 above).

The object to which I refer is the set of all sets.

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*[More will be said about economic value in connection with the "evolving necessities" invoked above, later on, under consideration 3: "The Boundaries of the Economy."]

The centrality of this conceptual object to the theory of sets is seldom explicitly acknowledged. The immediate reason for this is easily discovered. Though not numbered among the traditional paradoxes of Russell, et.al., the set of all sets is declared to be a "self-contradictory" object.[85] The discussions of this entity in contemporary writings on logic are typically rather clipped, the verdict virtually unanimous. Definitely is it regarded something to be avoided, excluded -- certainly not something to be enshrined at the heart of set theory.[86] Yet, conceptually, there it belongs. Given the set-theoretical way of defining things, the set of all sets is nothing less than the set-theoretical definition of the set-concept itself!

Here we require a word on "the set-theoretical way of defining things." The method of "extensional definition" lets stand for a predicate, the set of all objects which have the quality denoted by that predicate. The predicate "blue" is equated with the set of all objects having the color blue. The number two, or the quality "twoness" is defined to be the set of all "couples," or two-element-containing sets. The number three is defined to be the set of all triples, four the set of all quadruples, or "4-tuples," and so on. The concept of (cardinal) Number in general is defined by the set of all sets of n-tuples, n ranging over the cardinal numbers. That is, Number is the set of all numbers.

The "essence" denoted by a predicate-letter is expressed by the ensemble of entities which participate in that essence. The maximal totality of objects possessing a given quality expresses that quality as the only unity of, as the one commonality uniting, those objects.

Hence the quality "setness" should be represented by -- the set of all sets.

Let us call up this object in the mind, and observe its amazing properties. Actually, instead of "properties" with its static, nounistic connotations, perhaps "behavior" would be the better term. For the set of all sets, like the paradoxes of formal logic reviewed earlier, is a mental object which won't stay still in the mind, a "*mental* eventity"; a conceptual "process-object." It is inherently, *logically* dynamical. Its *being* is *activity*, ceaseless motion its only substance.

This can be seen as follows. The instant the set of all sets is constituted, it is less than itself. It is only the set of all *other* sets. That is, it lacks one member: itself. The *set* of all sets is, by stipulation, itself a set. Therefore *it* too belongs inside the set of *all* sets, as a *member* (element) of that set. The set of all sets must be a member of itself.

The obvious move, then, is to enter this "set of all sets except itself" into membership *within itself*. But this move does not dissolve the problem. It only reconstitutes the problem on a higher level. Adding a new member in this way, incorporating the set itself into itself, changes the set. This new set is not a member of the resulting attempt at "the set of all sets." That is, this new set is not a member of itself. Therefore, the attempt again fails. The would-be "set of all sets" still falls short of containing "*all* sets" by (at least) this one set, itself. Thus this new set must be self-incorporated. But the resulting set again fails to qualify as "the set of all sets" for the same reason, and so on *ad infinitum*.

Apparently then, the set of all sets can only be expressed by and *as* this *infinite movement*, this endless process of self-re-entry, self-internalization, and expansion-by-self. If we try to fix the set of all sets at any stage of this process, if we "stop" this movement at any "moment" and try to isolate that moment, we do not obtain our object. The set of all sets does not exist conceptually as any fixed, noun-like "object," but only as a *process* without end.

At every moment, the set of all sets is in perpetual disagreement with itself, that is, with its definition, its logical essence. This self-disagreement is the conceptual driving power of its logical self-propulsion, of its self-propelling "logic." Any particular, momentary embodiment in its sequential existence always falls short of its *being;* of "being" the set of all sets. Each attempted "appearance" of the set of all sets is a false appearance. Its existence at any moment always contradicts its essence. This self-disagreement or self-contradiction is expressed as a movement. It is expressed as an infinite progression of sets, or rather, as an infinite progression of "set-states" or "set-stages" of an ever-changing set which is ever becoming the set of all sets, without ever arriving at that entity as a final state.

It might help at this point to construct and scrutinize a formulaic, ideographic symbolization of this process called "the set of all sets." Suppose for the moment that we could construct a set which represents the initial attempt at the set of all sets as described above -- the "set of all other sets." Let S_o denote the entire content of that set. Then our initial set-state is

$$\{S_o\}$$

Let us recapitulate, using this ideographic device to illustrate the movement of thought. The problem was that $\{S_o\}$ does not contain $\{S_o\}$ itself as one of its members. Yet $\{S_o\}$ is a set. It is thus not the set of all sets, but, at best, the set of all sets save one; the set of all sets except itself. So $\{S_o\}$ must be brought into membership within itself. We have the movement:

$$\{S_o\} \rightarrow \{S_o, \{S_o\}\}$$

$\{S_o\}$ is a member of the resulting set $\{S_o, \{S_o\}\}$:

$$\{S_o\} \in \{S_o, \{S_o\}\}$$

However, the resulting set, for that very reason, is a new set, different from our initial one:

$$\{S_o\} \neq \{S_o, \{S_o\}\} .$$

And this new set is not to be found among its own members; is a set not included among the elements of the new attempt at "the set of all sets" (this attempt being none other than this new set itself.) Again, self-exclusion thwarts the attempt at constituting the set of all sets as a fixed object. This same deficiency is reproduced at a higher "type" level.*

Instead of a still, stable, completed set, we get a set-process, a set as process, for the extensional definition of "set-ness." At every stage, without cease, the set totality, beginning as $\{S_o\}$, keeps popping up inside itself as one of the elements. At every stage the ostensible whole "disappears into itself," yielding a new, higher type whole thereby:

$$\{S_o\} \rightarrow \{S_o,\{S_o\}\} \rightarrow \{S_o,\{S_o\},\{S_o,\{S_o\}\}\} \rightarrow \{S_o,\{S_o\},\{S_o,\{S_o\}\},\{S_o,\{S_o\},\{S_o,\{S_o\}\}\}\ \} \rightarrow \ldots .^{\dagger}$$

To Summarize: $\{S_o\}$, perforce of its inadequacy in denying itself membership, disappears as the "whole" set-of-all-sets and reappears as a "part" inside a new, expanded whole: $\{S_o,\{S_o\}\}$. This new whole differs from the previous one precisely in having one additional part or member: $\{S_o\}$. Since this new, second stage, in attempting to represent the set of all sets, includes $\{S_o\}$ as a member, $\{S_o\}$ can no longer be the whole. And so it is at every succeeding stage.

A Set-Theoretical Metaphor for Evolution

The set of all sets is an evolving set. It is not a merely cyclical process, like the Russell set.** The Russell set merely "pulsates," swallowing itself only to disgorge itself again in endless repetition. Thus that set is archetypical of gainless oscillation or "simple reproduction", a "circular" as opposed to "helical" process.†† But the set of all sets process is helical. There is an element of repetition, just as the helix preserves an element of the circle. But there is gain at every stage, as revealed by a number of standard set evaluators (set "state-variables" or measurements). Cardinality or "logical volume" (number of elements), and logical type,[†] for example, both increase by one unit with each step of self-incorporation. The set of all sets not only grows bigger with each stage of self-incorporation, but it grows in complexity as well. The basic content, S_o, is conserved, but moves into ever more ramified layers and textures of relationship to itself. As the set "grows," it grows ever richer in self-connections. For each new element added in this process is only a new variation or ramification of self-association of the same essential content: $\{S_o\}$, followed by $\{S_o, \{S_o\}\}$, followed by $\{S_o,\{S_o\},\{S_o,\{S_o\}\}\}$ and so on.

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*[Each self-incorporation raises the Russellian "logical type" of the resulting set by one type unit over that of its immediate predecessor.]

†[Note here the difference between $\{S_o,S_o,S_o,S_o,\ldots\}$ which reduces simply to $\{S_o\}$, and $\{S_o,\{S_o\}, \{S_o,\{S_o\}\},\ldots\}$ which does not reduce.]

**[The set of all sets which are not members of themselves. See p. 46 below.]

††[See pp. 38-39 above.]

Thus the set of all sets expresses a "Heraclitean"[*] progression of set-stages. In terms of the most basic, standard set metrics (such as cardinality and logical type), no two set-states can ever be the same. There can be no mere repetition of set-states. The set of all sets represents a process which is (i) "irreversible" or directional; (ii) noncyclical, and (iii) expandedly self-reproducing or evolutionary.

Note also that this set-process is propelled by "logic alone." That is, its own definition or logical essence is the source of its "evolutionary" energy. Thus the set of all sets represents self-movement. Its own nature, and the self-discrepancy inherent in that nature, is the cause of its logical dynamism. It is both the subject (initiating agent) and object (material worked upon by that agent) of i evolution or "development." It thus epitomizes reflexive process: self-evolution or self-development. It also epitomizes nonequilibrium dynamics; nonequilibrium as an inherent consequence of self-interaction. The immediate consequence of constituting any stage of, any particular attempt at, the set of all sets, is the disqualification of that set. This is a pure "boomerang" effect or logical, definitional self-effect of that set upon itself. The constitution of that set is precisely what disqualifies it. The "self-interaction" of each such set-state does not, as in an equilibrium ("fixed point") situation, confirm that state but rather negates it, moving the process away from that state. This kind of "self-interaction-propulsion" was exactly what we identified as the essence of non-linearity in dynamical systems.**

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[*] [Heraclitus' aphorism "You can't thrust your foot into the same river twice" evokes an image of Continuum as an ever-varying, never simply repeating flow. To that extent, the epithet "Heraclitean" aptly characterizes the quality of that "mental eventity" called "the set of all sets." But Heraclitus' aphorism also connotes a chaotic, orderless flux. The set of all sets, on the contrary, presents a highly ordered, progressive and expanding sequence of discrete states which i-

[†] [These changes are quantitatively insignificant however, given that $\{S_o\}$ is already infinite in cardinality and logical type, and $\infty + 1 = \infty$. We need different set-measures to adequately express the "helical" aspect.]

The "evolutionary" quality of the set of all sets process goes beyond the characteristics cited above. That process provides a conceptual, logical analog model, or "mental simulation" of a salient feature, a characteristic movement, of the process of evolution in external Nature.

Let us take up observation of the self-induced growth of the "set of all sets" from a vantage point imagined to be located as if "inside" this set. This set grows by self-internalization. What this means, seen from "inside" the set, is that new "particulars," new "individual objects," new elements "pop up" inside the set. New members arise inside and are added to its membership. But each new object or part(icular) happens to be a particularization and internalization of the totality, a "part-ification" of the whole as it stood in its immediately previous incarnation.

This phenomenology metaphorizes the pattern of concrete duration. It models the actual self-weaving of the space-time-mass-energy field of the cosmos, grasped as a single fabric. Or, in more current but more dualistic terms, it models the weaving of the fabric of cosmic mass-energy on the loom of universal space time. In the history of matter, new kinds of objects -- new qualities of material being -- emerge among the preceding contents of the universe as "self-invaginations," "self-involutions," or "infoldings" of the fabric of the space-time-mass-energy continuum, the "unitary field." For example, when the molecule first arises in a galaxy populated theretofore by atoms and subatomic objects alone, it arises as an

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*[continued from page 157. neither a simply continuous ("smooth") flow, nor a gainless, evolutionless turbulence. Thus an additional adjective -- perhaps "Cantorian" in reference to Cantor's ascending series of transfinite cardinalities -- is required for a more accurate description of the quality of the sets of all sets process.]

** [pp. 40-42]

"infolding" of the previous fabric or "field" of free atomic matter. The molecule is an atom (made) of atoms ("meta-atom" or "hyperatom"), an atom "of second degree." Likewise, later, the cell arises as an infolding of the fabric of marine molecules; as a "molecule of molecules," (a "molecule squared"); the meta-bion as an infolding of the fabric of cellular life; as a "cell of cells," and the (animal) society as an infolding of the manifold of metazoans, as an "organism of organisms."

The above description abstracts from the concrete details of the processes of "infolding," which are unique to each stage. But one feature is common to all these different stages as the general mechanism of "infolding." And this mechanism is profoundly analogous to the logical (definitional) or semantical-conceptual mechanism which drives the set of all sets process. We call it the process of "self-environment" (or "self-surroundment"; self-envelopment or "envelopment by likes").

Atoms originally arise in and into an environment of subatomic particles. Specifically, they arise when proton plasmas ("celestial Hydrogen gas clouds") become sufficiently proton-populous to densify and ignite as pure-Hydrogen stars.[87] But the consequences of the fruition and of the self-limitations of atomic evolution (stellar nucleosynthesis) -- starbursting novae and supernovae explosions -- increasingly populate the interstellar medium with atomic-level matter. That medium becomes increasingly a "gas" of atoms, no longer of "subatomic particles." As the process continues, a subtle transition occurs. Individual atoms no longer face immediately an environment of subatomic-level matter (free protons, electrons, etc.). Instead, they are "environmented" by other atoms. This "self-environment" or "envelopment by entities of the same kind" leads, once a "critical density" of atoms is achieved, to the accessibility of a self-interaction of the fabric of atoms. This self-interaction creates molecules, and molecular evolution. Similarly, the abundant success and inherent limits of molecular evolution at length produce a molecular self-environment and self-interaction which begets the first cells.[88]

Thence cellular proliferation arrives, at length, at a cellular self-environment and self-interaction that yields the metaphyta and metazoa -- "multicellular" biological individuals.

The crucial mechanism of these transitions, and the crux of the analogy with the set of all sets' driving mechanism -- is as follows. The "free energy" process or "metabolism" embodied by the *totality* of any of these stages of matter is qualitatively higher than the process animating and sustaining their individual elements. Each time, the whole becomes, at a certain stage of ripeness in its development, "more than the sum of its parts." That is, the organizational complexity or "negentropy value" of the whole becomes greater than the sum of the measures of the *individual* negentropies of its parts, measured as if they were in unorganized isolation from one another. This discrepancy, this surplus energy, drives the creation of a *new kind* of part.

Thus the ensemble of atoms *as they exist in the form of an evolving star* eventually constitute a process of free energy/negentropy utilization which transcends that represented by the same numbers and kinds of atoms measured individually, each as if a whole-unto-itself. Likewise, at a certain stage, the ensemble of molecules in the primeval ocean of Earth represents, *as a whole*, a negentropic process superior in quantity *and quality* to that embodied in an equivalent array of isolated molecules. The primeval* cell was precisely the precipitation or condensation *as a particular* of the molecular processes that characterized and were *distributed throughout* the whole primeval ocean before that condensation.

Later, photosynthesis, or the *auto*trophic cell, superseded the fermentative basis of the biosphere. Again, photosynthesis arose as an *internalization* into a multiplicity of particular, individualized objects--plant cells--of a "macro-metabolism" of "food"-molecule synthesis formerly distributed throughout the entire atmosphere-ocean system of the early Earth.

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*[heterotrophic, fermentative.[89]]

To summarize: the process of the totality of atoms, after a certain degree of populousness and maturity of atomic synthesis is reached within a star, is super-atomic. The totality of all these atom-processes finally amounts to something qualitatively more than atomic; to something "transatomic" in its lawfulness; in its behavioral process-potentialities. The whole made up of atoms is more than atomic in quality. The molecular quality is born first at the level of this "whole" formed by the manifold of atoms. This margin of qualitative difference between the whole and the part level becomes the free energy needed to effect the evolutionary transition. The newly emergent "molecular" quality of matter then becomes manifest as a multiplicity of new particulars: the molecules, the "self-reflexion" product of atomic evolution. The old whole becomes "reflected into itself" or "re-enters itself" as a new multiplicity of this new kind of part.

The resulting manifold of molecules becomes, in turn, as it matures in its evolution, transmolecular, protocellular, in the quality of "natural law" manifest in its evolving potentialities of behavior: It evolves to the creation of new laws, to transcendance of the laws and limitations of molecular behavior. It evolves a new dimension of phenomena, a qualitatively new "degree of freedom" in the sense that cellular matter is capable of activities and phenomena unknown among precellularly organized matter. And so again with the cellular manifold, and so on.

Collections of entities of one type, achieving sufficient multiplicity and maturity, become a new quality of process, a quality distributed throughout and defining the whole they thus form, which exceeds the process they represent individually and which "anticipates," or presages and sets up for, the next evolutionary type of "part."

Analogously, the collection of "all collections" creates a new collection which is *not among* the original collection. This new set is of a higher logical type than any of the others. To conform to the definition "set of *all* sets," this new set must *instantaneously*, immediately upon being formed, appear within, among them, as a particular, as part of the *content* of the "set of all sets" -- no longer as that "container" of them all which is not itself "contained." The whole that is more than the sum of its parts must become a part alongside the rest. The difference of the whole from the "sum"* of its parts *forces* this precipitation of a new *type* of part. This difference becomes manifest, at the level of the other parts, via this self-internalization or self-reentry; this infolding forming a new set or manifold of particulars containing a new part or element of a logical type level one higher than before.

The pattern described above is restricted neither to the main divisions of natural history nor to its pre-societal stages. This pattern is reproduced within sub-stages of evolution and, moreover, within the evolution of economies, i.e. of *processes* of economic reproduction. We will paradigmatise and harness this pattern, via the abstract image provided by the set of all sets process, for the economic modeling proposals to follow.

Sets as Operators: The "Heraclitean Product"

Suppose we carry out, in this realm of sets and logic, the same operation we have applied to the language of mathematics throughout this subsection. That is, suppose we treat sets too as "eventities" -- as active, process-entities, entities of the type described by "verbs" more than by "nouns." Suppose we conceive the set-symbol as an *operator*, standing for an operation. What insight might we thus obtain?

Interestingly enough, such a treatment leads to a reflexive function formulation of the set of all sets. The reflexive function is of the exact form which we derived originally on process-algebraic and operator-ideogrammatical grounds.

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*[Here sum = "collection" or "aggregate," i.e. "set."]

The set is traditionally conceived quite nounistically, as a fixed mental object, a static "collection." Quite naturally, the verbistic aspect of the set is expressed by the verb form of this one-word definition: the mental operation or process of "collec-tion." Viewed dynamically, a set is the process of collection of its elements.

This observation enables us to treat set symbols as operators. Such a treatment involves determining a meaning for the juxtaposition of set symbols--a meaning for "flexion" among sets. This meaning must be coherent with the "collec-tion" concept of the set, as well as with the established properties of sets. The "product" of the interaction of two sets appropriate to those criteria can be specified as follows [wherein $\emptyset$ denotes the "empty set"]:

$$\text{(i)} \quad \emptyset a \equiv \{\ \} \cdot a \equiv \{a\}$$

$$\text{(ii)} \quad \{a\} \cdot \{b\} \equiv \{a, \{b\}\}$$

$$\text{(iii)} \quad \{S_o\} \cdot \{S_o\} \equiv \{S_o\}^2 \equiv \{S_o, \{S_o\}\} \equiv \{S_1\}$$

For reasons soon to emerge, we dub this concept of set "multiplication" (or "product-ion") the "Heraclitean product" of sets. It can be described also as the "ingestion product" or "incorporation product" of sets. According to this concept, what a set does when interacting with another set or with *any* potential set-element is to ingest that entity, to include it--to incorporate it among the set's already-acquired elements. The "operation" characterizing or definitive of sets is this action of inclusion or incorporation. The null-set $\emptyset$ thus becomes the fundamental or "pure" form of the set operation; of the set conceived as a "collect*or*."

Let us see what happens under this concept in the case of set *self*-juxtaposition, that is, in the case of the "powers" of a set. For this purpose, let us select sets from the "set of all sets" sequence as our examples, starting with the first attempt, $\{S_o\}$:

$$\begin{array}{lllll}
\{S_o\}\{S_o\} = \{S_o\}^2 & = \{S_o,\{S_o\}\} & \equiv & \{S_1\} \\
\{S_1\}\{S_1\} = \{S_o\}^4 & = \{S_1,\{S_1\}\} & \equiv & \{S_2\} \\
\{S_2\}\{S_2\} = \{S_o\}^8 & = \{S_2,\{S_2\}\} & = & \{S_3\} \\
\vdots & & & \vdots \\
\{S_n\}\{S_n\} = \{S_o\}^{2^n} & = \{S_n,\{S_n\}\} & = & \{S_{n+1}\}
\end{array}$$

The pattern which emerges makes it clear that the set of all sets process can be formulated in terms of the self-interaction of the first-attempt set $\{S_o\}$, as a kind of dynamical reflexive function:

$$\{S_\tau\} = \{S_o\}^{2^\tau}.$$

Here $\{S_\tau\} \equiv X(\tau)$ stands for the (set-valued) state of the process at the stage τ. Thus τ, the independent variable here, does not stand for "time" in the sense of the continuous variable "t." Instead, it denotes a "discretized" time-like variable: the number of self-incorporations "elapsed," in the stage of self-reentry reached; the number of attempts.

Thus, from yet another angle--the ideographic, formulaic angle--we find the set of all sets qualifying as a "formal image" of our self-development paradigm for non-linearity in dynamical systems. The set of all sets can serve us as a mathematical metaphor, a logical simulation, a mental "kinetic model" for the paradigm of self-reflexive dynamics. Thereby it can provide a convenient, accessible, "mentally tangible" point-of-reference as we begin to apply this otherwise unfamiliar paradigm to the elucidation of heretofore puzzling phenomena of socio-economic life. The set of all sets process provides us, that is, with a kind of formalization of the concept of evolution here employed. If not yet a formal, axiomatic theory of evolutionary process, it at least provides an analogy, a logical analog model thereof. And this model suggests a theory of evolutionary causation and dynamics, such as we have just explored above, though one which has still to be fully and rigorously elaborated.

Of course, the shortcomings of this "formalization" and "formal kinetic image" of evolution, in terms of any comprehensive likeness are considerable and should be explicated here. They include the extremely schematic character of this analogue with respect to the repleteness, exceptionalism, and multifoliate variety of the actual course of evolutionary cosmogenesis,[90] and a certain kind of "second-order" repetitiousness. True, the set-analogue is "Heraclitean plus" (see footnote, pp. 157-8) at the "first-order" level. As we have stated, no set-state is ever repeated, and, moreover, each successor state has a cumulative, inclusive, and subsuming--i.e. an evolutionary--relation to its predecessor state(s). There is continuous novelty in the progression of set-states. But, observing at the "second-order level" the type of novelty is forever pre-ordained, predictable, and mechanically repeated. There is a qualitative novelty or change between each pair of set-states in the infinite progression. But this qualitative change is always of the same sort (or "meta-quality"). There is novelty, but there is no novelty of the novelty. The actual evolutionary process exhibits vastly more levels of "creativity" than this model.

Furthermore, the set-states appear to be discrete. The process appears to leap from one set-state to the next higher, apparently without any indication in the logic of the model of what happens in-between successive states; of what process(es) mediate the transitions. Only the sudden, qualitative leap is "visible" or "accounted for" in the model.

We might summarize these shortcomings by stating that the set of all sets is only a formal analog, an "idealization", of a self-reflexive evolutionary process--and only a most rudimentary, most abstract-schematic formal analog at that.

Apart from its analogic, metaphoric, exo-set-theoretical applications, what is this object, the set of all sets? What is it on its home ground, as an object of set theory proper? Early on in this exposition, in first introducing this object, we argued that it should be the very definition of the set according to the canons of set-theoretical, "logical" definition, the essence of "setness," and thus as the fundamental object of set theory; its "universe," its "field"--the "universe" or totality of sets.

Does it in fact work as such? In retrospect, after the "conceptual experience" we have accumulated since with this mental object, does this make sense? Does it fulfill the role just described and, if so, how?

The essense of the set is inclusion, gathering together--in short, the operation of collec-tion (and its result). The set of all sets, already begins from the result of an infinite* and apparently all-inclusive inclusion; an apparently unrestricted, unbounded, and comprehensive process of collection: the process which results in what we called "the set of all other sets" or "the set of all sets except itself."

But the set of all sets builds yet another infinite process on top of the first one. So essential is its drive to internalization, so relentless, and all-encompassing its logic of inclusion that, when nothing else remains that is eligible for collection, it "collects itself." This "self-collection" process unfolds as a second and interminable** stage of formation of the set of all sets. This property of eternally becoming itself (without ever arriving there) places it beyond the category of the still, fixed, equilibrium-like, or nonparadoxical entities of logic.

It is this very property, this anomaly of interminable dynamism, rendering the set of all sets "paradoxical" for classical logic, which makes it work as the set definition of the set, of the set quality in general. For this dynamism expresses the unrestricted, indiscriminate, exhaustive, and inexhaustible movement of inclusion--not stopping short of infinite self-inclusion as "last resort"--which is the essence of the "naive" set-concept."

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*[Observe that even getting to the "beginning" of the "set of all sets" process to the state $\{S_o\}$, the set of all other sets, is an endless process, since each and every (set-)element can itself be self-incorporated endlessly, creating a new element each time. In a sense then, the "set of all sets" process has neither beginning nor end. It has no beginning that can be arrived at by any finitary, additive, step-by-step procedure. Even its beginning is already "beyond infinity," e.g. "on the other side of" an infinite process.]

**[It is precisely this non-terminating property which is definitive with respect to the set of all sets process $\{S_\tau\} = \{S_o\}^{2^\tau}$. The Heraclitean product itself, though already expressing a "movement of inclusion," involves only a single, self-terminating

Evolutionary Operators

At this point, the "Heraclitean product" rule, introduced earlier, deserves special scrutiny.

First of all, why do we call it "Heraclitean"? We do so because of its role in the "dynamical reflexive function" formulation of the set of all sets. That is, we do so because of its implicit involvement in the expanding, cumulative, ever-changing, irreversible, never-repeating succession of set-states which is the set of all sets.[*]

Secondly, how does our Heraclitean product measure up in terms of those qualities with respect to which algebraic products are traditionally evaluated? In particular, let us measure this concept of set "multiplication" (interaction) in terms of three standard properties which such a concept will typically involve. The three properties are, of course, commutativity, associativity, and distributivity.

Mere inspection quickly reveals what may at first strike us as a startling fact: the Heraclitean product of sets transcends all three of these basic properties, which are axiomatic to the most familiar product, that of Real number arithmetic. Heraclitean "production" is not only noncommutative and nondistributive (hence nonlinear), but nonassociative as well! For example:

[**] [continued from page 166...step of inclusion -- vis. $\{a\}(\{b,c\}) = \{a,\{b,c\}\}$ -- a bounded, restricted, finite movement. It is the ("discretized") "dynamical reflexive function" form of $\{S_\tau\} = \{S_o\}^{2^\tau}$, as well as the value assigned to S_o, which gives this employment of the Heraclitean product its special significance.]

[*] [Actually, as earlier noted (see footnote pp. 157-8), the predicate "Heraclitean" falls short of comprehending the aspect of ordered everchangingness, of coherent, cumulative, growing, in short, of evolutionary progression which is expressed, with the help of the "Heraclitean product" rule, in the formula $\{S_\tau\} = \{S_o\}^{2^\tau}$. Another term -- "Chardinian product" for example -- might better stress this aspect. But we choose here to stress the non-oscillatory, not-simply-cyclical, non-repetitive aspect of the set of all sets which distinguishes it from the Russell set and the other classical paradoxes.]

> The higher kinds of number for the first time yield the concrete hope of placing the profound and subtle characteristics of bio-, psycho- and socio-transformations and processes on an adequate mathematical basis. Such kinds of number would thus introduce to the humanities their appropriate mathematics, which will not do them the grave and unscientific injustice of forcing them to fit some Procrustean bed of inadequate hypothesis or reductive definition. Man and man's sciences are now ready to go beyond the square root of minus one. With each new and higher kind of number a new and deeper algebra and arithmetic become possible, and hence a new and deeper functional analysis.[91]

Summary on the Set-Theoretical Metaphor

The set of all sets provides us with an additional image of our "self-development-by-self-action" paradigm -- an especially intense and vivid one. It does so from a quarter whence it might be least expected -- from the heart of the apparently static, timeless, equilibristic realm of set-theory and logic.

The set of all sets metaphorizes evolutionary becoming. It simulates self-expansion, and growth-without-end as qualitative growth; as a progression of qualitative changes. It has also been remarked (see footnote, page 166) that getting to the beginning ($\{S_o\}$) of the set of all sets process is as endless a task as getting to its "end." The appearance of a finite beginning is an illusion. Every set-element of any attempt at $\{S_o\}$ can itself be endlessly proliferated into new set-elements, also by endless self-inclusion. The positing of any set-element immediately implies that another set-element, formed by the self-inclusion of the posited one, is missing from the collection.

Thus the "beginning" as well as the "end" of the set of all sets is "at" the other side of an infinitary process. The set of all sets has neither beginning nor end in the finitary sense. Or, it is nothing but movement from beginning to end. Anything stopped or that stops is not it. Its substance is process only. It is a "doing," not a "being" in the prevalent, static and nounistic usage of the term. It exists only as the propagation of a definite activity, and ceases to be wherever that activity ceases. Stop it and it stops being. For it, beginning and end are not stopping points. It works as the set-defining set precisely because of this "non-stop" quality, and precisely by never reducing -- not at either "end" nor anywhere along the way "in between" -- to the finitary.

In view of these observations, the set of all sets might be better rendered by the formula*

$$\{S_\tau\} = \{S_{-\infty}\}^{2^\tau}$$

that by our original try

$$\{S_\tau\} = \{S_o\}^{2^\tau}$$

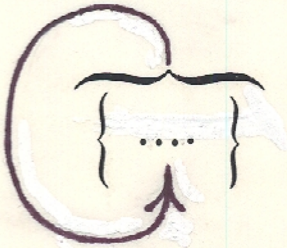
or, better still, by the apparently "stopped" or nondynamical, already-attained form:

$$\{S_o\}^{+\infty} \text{ or } \{S_{-\infty}\}^{+\infty}$$

which reflects the idea that, in the "timeless" realm of logic, the infinite self-incorporation process is called for immediately, is demanded to occur instantaneously.**

We have seen how it was the process approach, the "verbistic" or "operational" interpretation of the set, which led us to this new ideographic model of our paradigm. Grasping sets as operations -- in terms of the activity of collection -- and formulating a "product" or definition of juxtapositional behavior accordingly, led directly to a "dynamical reflexive function"-like formula for the set of all sets.

The essential movement of the set of all sets can be summarized pictorally:



[Figure 11: Set of All Sets Pictograph]

**[This requirement that a process involving an infinite number of steps take place, timelessly, instantaneously implies a kind of "time within instantaneity," and might be used to develop a set-theoretical, logical model of time. The exploration of this conjecture is, however, outside the scope of the present paper.]

*[Note also that 2^τ is the formula for the cardinality of the "power-set," or set of all subsets, of a set of cardinality τ.]

✓ BG
10 Sep. 1992

The infinite contents within the two vertical braces are, for every stage, indicated entirely by ellipsis (...). The span of the superscripted horizontal brace indicates that the whole entity underneath -- braces and all -- is swept up and re-entered into its own *content*, inside the vertical braces. This self-entry, as indicated by the arrow, goes on continually.*

The "dynamical" interpretation of the set concept, which led to the reflexive function formulation of the set of all sets, may also hold the key to clarifying other longstanding conceptual anomalies of "naive" set theory. Prime among these are the "mysterious" elementary set relations involving the empty set(s) and the "singletons" or unit-sets (sets with but a single element):

(1) $\{\{\{\ \}\}\} \neq \{\{\ \}\} \neq \{\ \} \neq$, i.e., in our notation, $\emptyset^3 \neq \emptyset^2 \neq \emptyset^1 \neq \emptyset^0$
or, in general, $\emptyset^{n+1} \neq \emptyset^n$;

(2) $\{a\} \neq a$, i.e. $\emptyset a \neq a$, and $\{\{a\}\} \neq \{a\}$ and so on (in general $\emptyset^{n+1}(a) \neq \emptyset^n(a)$);

(3) $(\forall \text{ set } S)(\emptyset \subseteq S)$ & $\sim(\forall \text{ set } S)(\emptyset \in S)$.

The above relations appear paradoxical for the notion of sets as *mere* collections, mere aggregates of their elements. This notion would imply that a set is *nothing but* the "gathering" or mental association of its contents; is nothing *without* or apart from that content. But the empty set is a set, and different from "pure Nothing" -- $\{\ \} \neq$ -- even though its content is "pure Nothing." The set with only one element is different from that element. The first two relations call into question the meaning and function of the brackets or braces in set-notation. This notational issue, in turn, calls into question the *idea* behind the ideogram. It raises the problem: Is a set *as a whole* something more than the mere collection of its elements, more than just the (inhomogeneous) "sum" of its "parts." The two special cases of the empty set(s) and the singletons are important here only

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*[This depiction brings out a possible conceptual connection between the set of all sets' self-re-entry process, and that of the toroidal-vortical or "cyclonic" flow-field so often encountered in hydrodynamics. Recent work points to this cyclonic form as the long-sought three-dimensional generalization of the *soliton solutions*. The *soliton solutions* are the recently discovered class of *exact* solutions to a wide variety of nonlinear *partial* differential equation models encountered in hydrodynamics, magnetohydrodynamics, and nonlinear wave theory generally. See PO & LC, *Appendix II*: Cyclones and Solitons.]

because they happen to be the extreme cases where this problem shows up most visibly and vividly. The problem, however, afflicts all sets. What is the difference between the set as a single, "singular" entity, as a unity, and the set as a "plural," a plurality of elements?

The first two examples above seem to impute a substantiality[*] of some kind to that aspect of sets denoted by the braces. The braces seem to function as a kind of "container," separate from the contents, the mere collection or "sum of elements/parts" which they "contain." The "container" is evidently something, not pure nothing, because it can stand alone as a set-entity: the empty set is different from nothing. The braces ("containers") constitute an entity distinct from their content to such an extent that they form a bonafide set even when their content is nothing, no element(s). The container alone is enough to make a set. Moreover, a container containing a container containing nothing, {{ }}, is different from a single container of nothing { }, and so on. Likewise, a set containing a single element "a" -- {a} -- is different from that element by itself, "unsetted"; un-contained. A set is, then, evidently something more than a mere collection. It appears to be the collection or ensemble of its elements plus the "container" which collects them and keeps them collected; which "holds them together," and which, like a cell membrane, separates them from the rest of the Universe of elements.

But what is this "container"? What is it "made of"?

The complexity involved in this question is indicated when it is remembered that these "containers" or "membranes" can crisscross, intersect, overlap, and interpenetrate endlessly. That is, sets can intersect. Any given element can be included in many different sets simultaneously. Besides this ability to pass through itself, the "substance" of which the "containers" are "composed" is ~~if~~ of a different kind than that of the elements. The container is not "substantial" on a par with

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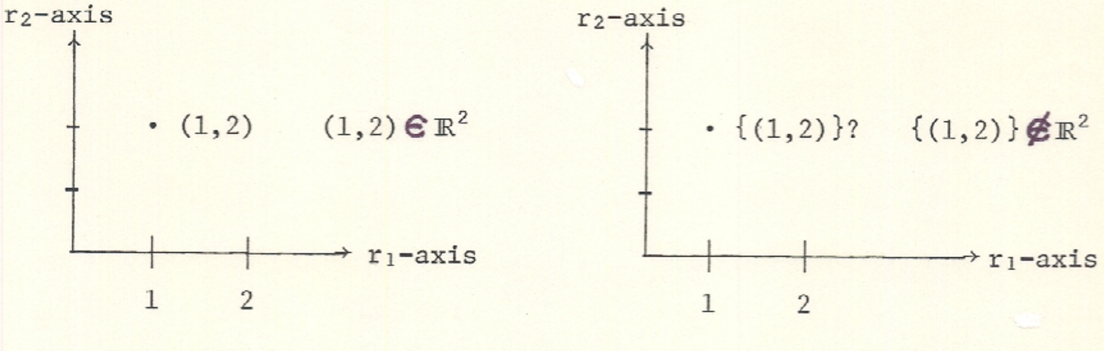
[*] [The order of materiality here invoked is not that perceived by the external senses, but only by the mind's eye, by looking inward. Concepts are objects vis-à-vis the thinking subject, albeit of a subtle sort, subsisting only in the special ambience of the internal space of the mind. But any kind of objectivity implies some kind of materiality, some order of substance.]

the elements. The container is not itself an element or of the same order of substantiality/materiality as the elements. And yet it appears to exert a decisive influence on the properties and nature of sets; to be an integral part of the set *concept*.

Consider the point (1,2) in $\mathbb{R}^2$. Now consider the point-*set* containing (1,2) as its only element -- $\{(1,2)\}$. Now clearly $(1,2) \in \mathbb{R}^2$, $\{(1,2)\} \subset \mathbb{R}^2$, but $\{(1,2)\} \notin \mathbb{R}^2$. How would we *depict* analytic-geometrically the difference between (1,2) and $\{(1,2)\}$?



(A) (B)

[*Figure 12A&B*: Single Points as Elements vs. One-Point Point-Sets]

The set-constituting "container" of the point (1,2) is not of the same "material" of which the rest of $\mathbb{R}^2$ is composed -- it is not made of "points." How, then, can this "container" be depicted?

Perhaps the "container" is not a static being or substantial "object" at all. Perhaps the braces represent the staticized image, the conceptual traces, of the mental act of distinguishing the set-content from the rest of the Universe of elements; of the conceptual *operation* of set-formation, of the mental *process* of "collec-tion." Constituting a set is a *mental* act of *recognition* of elements that were "already there" before this set-forming act, but not *as* a separate, distinguished, *out-standing* collective entity, standing out against the "background" of the rest of the "space" or Universe of elements. Set-constitution is the transformation of a neutral undifferentiated space into one with foreground/background contrast in terms of the play or distribution of *mental attention* with respect to that space.

The apparent "substantiality," "materiality" or "object-ivity"* of the set "container" or "membrane" vis-à-vis "set-substance" ("element-substance") would then turn out to be a delusory appearance. This appearance would be a conceptual artifact of the static-ized, nounified interpretation of set processes. The meaning and behavior of the braces in set notation would be explained as follows. The braces would be comprehended as registering the conceptual traces, the frozen vestiges, of those movements of mind involved in set-cognition; as the notational shadows of the mental activity of "(s)election" (to use the Boolean term)[92].

In certain cases of set-spaces, the traces of this differentiating or distinguishing process and of its agency, the cognizing mind, can be depicted in terms of the same "substance" of which the element-space itself is composed, and thus as a part of the space. This depictability is connected to the concepts of "closures" and "boundaries" of "open" and "closed" point-sets in "dense," continuous point-spaces.

It is commonly misconceived that, for example, a point divides a line into two distinct point-sets; likewise that a closed line (simple closed curve) divides a plane into two non-overlapping point-sets, thus representing a procedure of dichotomy. Actually, this procedure is a trichotomy. The point divides the line-universe into two ray-sets plus a third set: the singleton of the dividing-set itself. The closed curve divides the plane-universe into three sets: the interior, the exterior, and the closed-curve boundary-set itself. Of course, the "borderline case" of the boundary-set is always of one dimension inferior to the other two sets which it creates and separates. Hence, the boundary set -- the geometric counterpart of the formulaic braces-symbol -- is of a different topological order of "materiality" vis-à-vis the other two, dichotomized sets (and thus often used to be overlooked in a typical high school geometry course's naive account).

This geometrical phenomenon of "trichotomy" represents a mathematical, "mental-operations" version and model for the general scientific/experimental phenomenon of "operational uncertainty." The agency of experimental action/manipulation

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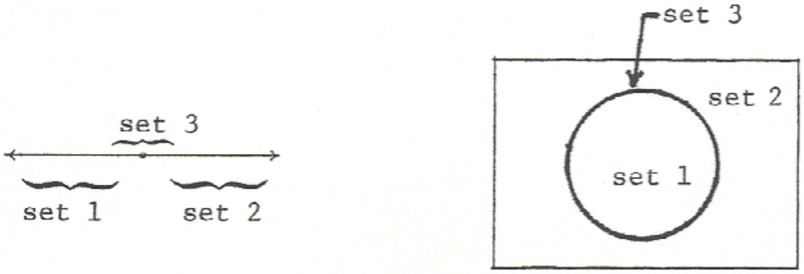
*[Object-ness in the static, nounist sense, not the "eventity" sense.]



[*Figure 13*: Geometrical Symbols Corresponding to the Braces of Formulaic Set Notation]

becomes *part of* the *result* of that action/manipulation; is ingredient in the resulting "observation." The action of observation is partially constitutive of, is "etched into," the outcome of the experiment. The action leaves its mark in the object of the action. The agent of action leaves its impression in the material upon which it works. Here, the activity of division casts its shadow *permanently* into the universe of discourse it divides, mirroring and statically recording the mental activity of attention in selectively *recognizing* and singling out the sets 1 and 2 above. The mental operation of dividing the point-Universes above "in two" leaves its mark *within* these point-Universes which are the objects of this dividing action.

The problem of depicting the empty set(s) and unit sets or singletons in $\mathbb{R}^n$ (n=1,2,3) still remains, but in a slightly different form. No boundary made of points of $\mathbb{R}^n$ can properly separate a one-point-containing set from the rest of the $\mathbb{R}^n$ space of points. The point itself must be its own boundary-set or "container," as well as its own "contents." The very act of "highlighting" or emphasizing a single point -- of making that point *stand out* against and *contrast with* the rest of the points of the space as background -- must itself already denote the formation of the set containing just that point. Thus it is points as *elements*, not points as sets, which are undepictable in this mode.*

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*[The foregoing discussion brings to the fore certain anomalies in the concept of "point" itself which, however, we will not take up here.]

The empty set would have to be depicted, in this mode, as a blank area; as the whole space "unhighlighted" or "unemphasized" (including even the axis-sets). The universal set, by contrast, would have to be represented as a completely darkened area; the whole area representing the space would be "highlighted" or "emphasized."

Not only the set of all sets, but even such apparently "simple" entities as empty sets and singletons, present paradoxes in terms of the manner in which naive set theory currently accounts for itself. The light shed by the "process" or "eventity" approach toward resolving these conceptual anomalies is a further testament to its potential for resolving longstanding problems. This "verbistic" or "operational" approach to the "ideogrammar" of mathematical ideation and of mathematical ideography goes back at least to Boole. It holds that ideograms represent ideas as mental processes; mental operations; mental acts, mental events or "eventities." Formulas written with ideograms are conceived as ideographic "verbs" or better, as "noun-verb unities" signifying, when in imperative mode, a "program" of commands for the calculative acts to be performed by the mind, just as the notations of computer programming languages represent the operational orders transmitted to an electronic calculating device. In declarative mode, a formula composed of mathematical ideograms records the mental calculative operations necessary to arrive at certain results, "as if" already performed.

Set theory is widely considered today to represent the very foundation of mathematics and, sometimes, of logic as well. The set-theoretical ramifications of the "verbistic," operatorial approach to "mathematical linguistics"* in the case of the set of all sets, as well as in the cases of singletons and empty sets just discussed, will inform and find frequent application in the economic modeling uses of mathematical language which are to follow.

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*[By "mathematical linguistics" is here meant, not the mathematics of or the mathematical approach to linguistics, but the linguistics of mathematics; the results of a linguistic approach to and analysis of mathematical ideography.]

✓28 Nov. 1992 BG

Considerations on Mathematics: Conclusion
(The Reflexivity Paradigm)

Throughout this subsection we have been marshalling metaphors, gathering together a wide variety of verbal, diagrammatic, formulaic, logical, geometric, and set-theoretical images of a new and fundamental paradigm of process. We have been preparing to apply this paradigm to an explanation of that peculiar combination of growth and crisis experienced in economic life. The paradigm is one of self-development; of evolution driven by self-reflexion, self-reflexive action.

This preparation has aimed to be more than a mere collection of such images however. True, the presentation of a wide variety of models or conceptual examples of our paradigm in itself helps to overcome its unfamiliarity. This multiplicity enables one to distinguish the common, essential features of the paradigm from those which belong to special cases only. But we aimed also at a Platonic process, at an intrinsic dialogue building the concept of self-development via a succession of evolutionarily-ordered stages. The objectification in language of each successive example of the paradigm, whether in graphic, ideographic, or phonetic symbols, was aimed to precipitate an improved or intensified re-subjectification of the paradigm -- the formation of a more advanced concept of reflexive development -- in preparation for assimilating the next level of example, and so on. Our exposition thus aimed at the self-development of the concept of self-development, so that even the experience of forming the concept would second the concept's content.

We have also developed the expanded symbolic or linguistic medium which appears prerequisite to the analytical treatment of nonlinear -- that is, self-reflexive -- processes. This medium is the "process-algebra" and "process-geometry" -- the dynamic, anisotropic analytical geometry of $\mathbb{H}$ and the "dynamical arithmetic"* of the hypernumbers.

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*[By "dynamical arithmetic" is meant an arithmetic (and, therefore, its algebra) involving functions of time -- timed-values, or timed-variables -- and therefore, by implication, the integral and differential operators with respect to time (the hypernumbers σ and σ^{-1}) which operate informatively on such functions. Thus "dynamical arithmetic" and "dynamical algebra" include all of mathematics up to integrodifferential equations. State-space provides the "dynamical analytical geometry" corresponding to this arithmetic and algebra.]

Building a Mathematical Language

The guiding method throughout this subsection has been the "linguistic" analysis of mathematics, and the "linguistic" approach to the foundations of mathematics. Considerations of notational "ideogrammar" have provided clues to the new mathematical concepts needed to resolve mathematical difficulties of long-standing, principally the difficulties with nonlinear differential equations.

In the perspective of this method, mathematical work is neither an exercise in formal logic, nor an exploration of the logical implications of arbitrary premises, nor a direct apprehension of Eternal Truth. Rather, mathematical work is a beginning of conscious creation of human language, of conscious work upon that most fundamental of human technologies. In terms of cultural history, such heightened consciousness about language has shown itself to be a prime tendency of the modern period in other, nonideographic areas as well (the recent developments of the "language analysis" tradition in philosophy and of the field of "linguistics" as the study of phonetic language are signs of this tendency, however successful or abortive they may be adjudged). But, because of the peculiar placement of the organ of mathematics in the organism of the various sciences, this conscious, deliberate evolution of human language in its ideographic variant is also the study of the human mind, and a continuing development of new concepts and of new, higher conceptual powers. The intentional development of ideographic language cannot be sharply separated from the study of ideas and their development. This is so even though, historically to date, mathematical work has been largely, unnaturally, restricted to a special kind of ideational development (to "linear logic" and linear, especially cardinal, mathematics). But in essence, if only sporadically so far in its manifest historical existence, mathematics involves the scientific, and rigorously experimental exploration of the so-called "subjective," or internal-objective, dimension; of the "internal space" of the mind and the discovery of the laws of its constitutive movements or processes. Mathematics is a "linguistic" science, therefore a science of "subjectivity" as well as a science of (external) objectivity. For language forms an interface between the "subjective" and objective realms. But mathematics is not just "linguistic" -- not just ideography. Mathematics is *ideometry by means of ideography*. By "metry" in "ideometry" we mean the "measurement" or *assessment* of ideas -- any and *all* ideas -- in the fullest sense, "qualitative" as well as "quantitative."

Traditional treatments of ideographic logic and mathematics emphasize their aspects as *systems of truths*. Traditional treatments of phonetic natural languages emphasize their aspects as *systems of descriptions*, and downplay their aspects as systems of truths. Yet Whorf, for one, has made us aware of the degree to which natural languages are permeated with ontological commitments and epistemological assumptions in the very depths of their grammatical substructures.[93] So we have presented mathematics as a system of description. We have shown how undue restrictiveness with respect to the "system of truths" aspect, especially when it involves *unconscious* ontological commitments, such as those we have termed "nounism," can lead to descriptive difficulties, actually aborting the linguistic potential of mathematics -- viz. in regard to exact description of the state-space trajectories of *nonlinear* dynamical systems.

This linguistic approach to mathematics explains the title of this subsection: "Considerations on Mathematics *as a Tool of Description*."

The consideration conducted up to this point can best be characterized as a work of *linguistic engineering*, concerned with constructing a language with descriptive facilities adequate to the economic modeling we wish to pursue. The hypothesis introduced in section I, connecting economic crisis, technological change, and credit/monetary processes, is what we wish to model, and, later, test with data from "live" economies to determine how well the model fits the reality. How then does the reflexivity paradigm, and the hypernumber reflexive function *language* for describing self-reflexive, self-developing process, aid us in our economic modeling task?

Obsolescence Depreciation as a Reflexive Effect

We already intimated an answer at the outset of this subsection, on pages 36 to 38, where we introduced the concept of "economic reproduction." The output (product) from the previous annual production cycle of an economy forms a central part of the input (material of production; "means" of production in the widest sense) to that economy in its next annual production cycle. This formulation provides one way of seeing the more "delayed" aspects of the manner in which the interaction of the economy with itself is mediated. This formulation encompasses the most general, complete answer regarding the "self-interactive" nature of the economy, subsuming many particular modes of self-interaction. But here we want a more specific answer, one which applies our ideographic language for self-interaction directly to the heart of our interconnection hypothesis.

That answer is as follows: Technological mutation, innovation in the means, methods, and organization of production, leading to heightened productivity, is an inherent, inseparable feature of capital accumulation. Capital accumulation is not just quantitative but also qualitative change (growth) in the capital stock of society. Thus the obsolescence depreciation effect at the heart of our hypothesis is an intrinsic self-effect of capital accumulation. Capital only exists, or continues to exist as capital, when it exists as continually profitable capital, i.e. as accumulating capital. Therefore, we might as well say that technological depreciation is a self-reflexive action of capital (period). Since capital exists as such only when it is "in" this movement of accumulation, "capital" properly conceived IS this movement. The movement of accumulation is the essential activity of capital; integral and definitive of its "being." Thus obsolescence-depreciation or productivity-growth-caused devaluation of fixed capital is a reflexive function of capital itself. It is the self-depreciation of capital, highly amenable to dynamical reflexive function formulation. The older portions of accumulation are constantly being devalued by the newer "buds" ever emerging from those older parts themselves. The older portions of the capital plant of society are continually depreciated, by more than the loss in value attributable to physical wear and tear, by the productivity-effects of newer plants formed from the orchestrated output of the older stock itself.

If our hypothesis is correct, this self-reflexion of the capital-process is the central cause of economic crisis for industrial capitalist type economies.

We shall see later, however, that the more general picture of the "self-reflexive" character of social economies (pp. 36-38) subsumes this particular self-reflexive effect, and links the specifically industrial-capitalist type of economic crisis to the types of crisis manifested by other institutional forms of social economy.

A Physical Example of Self-Reflexive Process

In closing these considerations on mathematics, wherein we have introduced the reflexivity paradigm primarily in terms of mental, conceptual models, it would be helpful to provide a nonmathematical as well as noneconomic example of a self-reflexive process, especially a physical example, to afford a more concrete image of the kind of phenomena which this paradigm addresses. Of course, in the truest sense, all images, all models, are conceptual models. But the conceptual models of "self-reflexivity" and "self-developing process" offered so far have been "abstract" models, that is, models referred to internal, conceptual, mental objects and processes. The state-space solution-geometries of nonlinear differential equations, the paradoxes of formal logic, and the set of all sets were objects of this kind. Here we want a conceptual model which is referred to the "concrete" and external, to objects of sensory perception. (Nonetheless, the linguistic, descriptive power of mathematics for natural science and engineering resides in the connection between the two kinds of conceptual models; in our ability to map, mimic, and metaphorize the latter, external phenomena by means of the former, inner, mental ones.)

Our physical example of reflexivity comes from the heartland of physics, from the dynamics of charged particles. It may at first appear experientially inaccessible, physically intangible in effect. This is because of its microscopic scale.

But further consideration will demonstrate otherwise.

Moving charged particles source "fields" of "electric-magnetic force." Our example concerns the forces which these fields exert back upon their sources, the particles themselves. Here it is the same as in the case of other reflexive phenomena, such as differential equation nonlinearity, and logical paradox. The reflexive phenomenon of "self-fields" and "self-forces" centers a zone of unresolved conceptual difficulty in physics. And in this case, the difficulty prevails despite the fact that the classical mathematical model of the phenomenon is technically *linear*. Again, it emerges that conceptual blocks with regard to reflexivity, rather than any merely technical matter in the narrower sense, underlie the chronic lack of success in resolving the difficulty:

> If a particle radiates energy in the form of electromagnetic waves, it must suffer a damping to its mechanical energy to pay for it.... The nature of this damping force is very strange. Unlike other forces in physics, it depends on the *second* derivative of the velocity, i.e. rate of change of acceleration, a fact which will prove of great significance. ... what is the origin of this damping force[?] Thus far, we have only been dealing with fields generated by special types of given source distributions. The picture is not complete, however, unless the effect of the fields back on the sources is included... [the complete] field is the *total* field incident on the ... particle *including the field of the particle itself*. It is this self-action which allows a charged particle to radiate energy in the absence of all other charges or fields.... self-energy divergences [infinities]... arise from the interaction of a particle with its own electromagnetic field....Such divergences have plagued the subject of both classical and quantum mechanics for many years. Their removal is at present only a formal device, called renormalization, based on the fact that observable quantities always turn out to be the finite difference between two diverging [infinite] quantities. The infinities are therefore merely subtracted out and ignored.[94]

> 'One ends up with the Lorentz equation of motion' -- but only after hurdling problems of principle along the way. One would understand what a particle is if one understood how to do the calculation of balance of energy-momentum with all rigor. Few calculations in all of physics have been done in so many ways by so many leading investigators... Among the issues that develop are two that never cease to compel attention. (1) The particle responds according to the Lorentz law to a field. This field is the sum of a contribution from external sources and from the particle itself. How is the field exerted by the particle on itself to be calculated?...[95]

The terms in the classical equation of motion which describe this self-interaction of the particle mediated by its own electric-magnetic field, all involve a statical reflexive function of degree two, a self-operation of the charge e: $e(e) = e^2$. But the charge "operator" here is a Real-number constant -- neither a function of time nor an unknown value. Hence, despite its inclusion of these "reflexions" or reflexive terms, the equation of motion remains linear.

Our example does not end with these one-particle models, however. The microphysical phenomenon of single-particle self-interaction is ingredient in the dramatic macroscopic phenomena of *plasmas*. A plasma is a seething sea of charged particles, particularly of charged *atoms*. A plasma is a gas superheated to the point of qualitative breakthrough into a new phase dimension, behaviorally beyond the solid, liquid, and gaseous phase qualities. This new dimension of material behavior is based upon the electrical *deneutralization* of matter, the high-temperature dissociation of atomic electrons from atomic nucleii -- the comprehensive *ionization* of the atomic elements composing a (former) gas.

Plasmas could be described as "hyperfluids." They are huge, macroscopically conspicuous "collections" of microscopic, interacting and self-interacting charged particles, fused into a behavioral unity, behaving as a cohesive, coordinated system. Not only is the phenomenology of the plasma phase of matter replete with dramatic surprises vis-à-vis the expectations of linear, irreflexive *theory*. Comprehension of this phenomenology is presently of the most urgent *practical* importance as well. Such comprehension holds the key to harnessing nuclear fusion as the solution to the world's energy problem.

The plasma phase is a seminal example of a self-reflexive process. At any given "freeze-frame" moment of its existence, a plasma appears as a definite spatial distribution of its charged-particle content. It presents an elaborate texture of varying local densities and compositions. This momentary distribution implies a definite field-geometry throughout and about the plasma. This force-field geometry in turn acts upon the particles, moving them, changing their distribution. This changed distribution, in turn, changes the field-geometry, which changes the distribution again, and so on.

The spatial organization -- or distribution -- of the plasma determines its overall field. But this plasma field, in conjunction with any external fields, determines the plasma's spatial organization. We have a "self-distribution" continually re-distributing itself in space as time elapses. We have a "circular" process of mutual or "inter-"determination of field-geometry and plasma particle-geometry. Plasmas thus exemplify *mediately self-determining* process-entities. A plasma is a self-moving, self-shaping, self-changing entity -- a "self-forming content." It is -- partially, apart from external fields, container contours, and other environmental influences -- a self-determining process. But it is not *immediately* or instantaneously self-determining. The self-shaping of the plasma-geometry is mediated by the plasma-field.[*]

Given the volatility suggested by this self-interactivity, it is no surprise that even "stable" solutions of the nonlinear model equations of plasmas describe motions, not static configurations. Of particular interest are the "self-consistent motions," defined as follows:

> A simple way of describing the motion of a system of particles is to consider the motion of each particle in the field of the others. This field is self-consistent if the motion of the particles determined in the field leads to a charge distribution which gives rise to the same field.[96]

That is, a self-consistent motion is one which reproduces the field which reproduces the motion which reproduces the field... a *self-reproducing* flow of plasma. Now, as V. N. Tystovich observed in an important recent book, *Theory of Turbulent Plasma:*

> The equations of motion for a plasma are nonlinear. It is therefore possible, at least in principle, to have self-consistent nonlinear motion.[97]

One of the primary candidates for such "nonlinear self-consistent motions" are the still little-understood toroidal eddies or *cyclonic* vortices which arise both in experimental practice and in numerical solutions of the plasma equations.[98] Neutrons, apparently of fusion origin, are sometimes detected emitting from the "eye" regions of these cyclonic whirlpools of plasma. This indicates that the compression, heating, and confinement taking place in that maximally-constricted region of the vortical flow creates conditions inducing the fusion reaction. In that case, the plasma would be a "self-changing" process in a yet stronger sense, altering the very atomic composition of the medium of which it consists, transforming Hydrogen nucleii into Helium nucleii.

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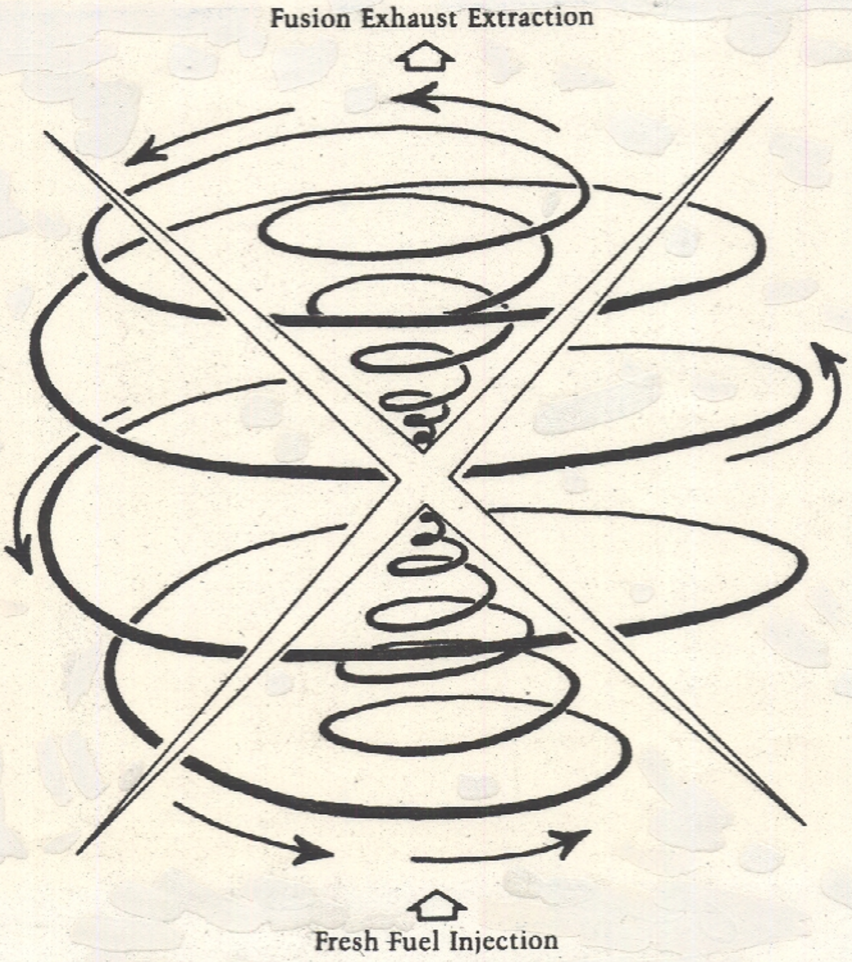
[*] [However, no absolute, radical dualism between "particle" and "field" is conceptually tenable (see Einstein and Infield[99]; DeBroglie[100]). Indeed, theories are emerging which explain particles entirely as nonlinear waves in a unitary field (see U. Enz.[101]).]



The "Plasma Bottle." The toroidal vortex is not a simple vortex, but one joined end to end, re-entering itself; a vortical flow "rolling" upon the surface of the torus. Such plasma flowings apparently represent "self-consistent" nonlinear motions, in which charged particles of the plasma move according to their own field, creating a charge distribution, which gives rise to the same field once again.

Figure 14

We have described the self-reflexive character of plasma activity. This character already makes plasma a self-developing entity in the sense of physical geometry -- of its shape and spatial texturation. But this is a superficial sense relative to that just cited above. In that latter sense a plasma may, mediated by the vortex structures it spontaneously creates, "work up" or "develop" its own content. It may act, in short, as a "self-developing medium" reflexively raising its own substance to a qualitatively higher level.

But how do we model such a medium mathematically in explicit congruity with our reflexivity paradigm? The answer is that we would want, for the state function of a plasma, $S(t)$, a reflexive function of the following form:

$$S(t) = g(t)(\mathsf{P})^{f(t)} = \tilde{S}(t)(\mathsf{P})^{f(t)} \quad . \qquad \text{(I)}$$

In (I), $g(t) = \tilde{S}(t)$ denotes "the state of the rest of the Universe" as environment to the plasma, i.e., formulated "from the point of view of the plasma," in terms of how it impinges upon the plasma. As usual, the exponentiated function-value, $f(t)$, incorporates the lag-time or "mediation interval," in this case of the plasma's field-mediated self-interaction.*

In P, the "plasma self-operator," however, we encounter a mathematical object of a new kind. The state of the plasma at any time t, $S(t)$, is the product of the environmental state $\tilde{S}(t)$ and the self-effect of the plasma, $\mathsf{P}^{f(t)}$. The values of $\tilde{S}(t)$ must, in general, be mathematical objects of the same kind as P, in order for the operators to be compatible, i.e. in order for the "product-tion" or "multiplication" (flexion) $\tilde{S}(t) \cdot \mathsf{P}^{f(t)}$ to take place. Similarly, the result of that flexion, S(t), must, in general, be valued in mathematical objects of the same type. Thus the entire state-function as expressed above is implicitly written in terms of this new kind of operator.

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*[The self-interaction of the plasma via its field does not constitute an instantaneous "action-at-distance." The "velocity" of self-interaction is limited ultimately by c, the velocity of photons, even though the "distance" traversed by this "action" carried by the field-mediating photons might at first sight appear to be zero, since the action of the field "comes from" and "goes to" the "same place," the plasma.]

Below we describe this new kind of operator, the specifications it must meet to make workable state-descriptions of the form of the above formula. Here let it be said that the work to implement this new kind of operator is still going on, and that is one major reason for the preliminary and merely preparatory character of this paper. For this is the kind of operator needed for the fulfillment of our economic modeling programme. It is not a foregone conclusion, though it appears increasingly likely, that it is possible to implement fully all of these specifications in a single operator, given the means at hand. This kind of operator is "internally" the most complicated yet. But success at implementation would make for a much simpler, more direct, much less opaque relationship between reflexive processes and formulaic models like (I) above. The connection between external *processes* and the ideographic modeling *language* would become virtually *mimetic*, rendering the writing and "reading" of mathematical models much more readily learnable.

In (I) above, P stands for the *whole plasma* -- its initial state. But P is not a state-vector, a finite list of state-variable-values. P is meant to stand for the *whole geometry* of the plasma, describing the plasma in a very detailed fashion, including "*geographical*" (physical-spatial) detail. To make (I) work, P must stand for a "multi-geometric variable," an *array of* "geometries." These "geometries" include the "manifolds" or contoured (hyper)surfaces which describe the temperature, charge, density, etc. distributions of the plasma in physical space. They include also the "manifold(s)" which describe the *dynamical laws* of the plasma -- that is, which describe the trajectorial "fields" or "textures" of the state-space(s) needed to describe the plasma in terms of its "aggregate," truly "holistic," non-geographically-distributable, state-variables. Moreover, this "array of geometries" must be so constituted that it represents an *operation*, that it, and the "geometries" composing it, function as operators. P operating upon P must produce the new array of geometries which describes the plasma at time t_2 such that $f(t_2) = 2$. Likewise, the computational interaction of the "manifolds" arrayed in P^2 must be such that $\mathsf{P}^2 \cdot \mathsf{P}^2 = \mathsf{P}^4$ describes the plasma at time t_4 such that $f(t_4) = 4$, and so on. What does this mean in detail, and how could it be arranged mathematically (notationally and conceptually)?

Manifolds as Operators

The manifold meanings of "manifold" in mathematics form one of that subject's richest, most pregnant ambiguities.[102] All of them fuse, without confusion, in the idea behind an ideogram like P in (I) above.

I. Most broadly, a manifold is "many-fold," is a "many-fold." It is a single entity composed of many -- of a variety of -- parts. In this sense, "manifold" connotes a set.

II. Geometrically, a manifold connotes a contoured, curved surface or hypersurface*, a "surface of many folds." Such "curvacious" or true "manyfolds" stand in contrast to planes and hyperplanes -- the "rectilinear," "flat," or Euclidean (hyper)surfaces; the "manifolds" without any "folds." A true "manyfold," a curved or nonrectilinear surface, is non-Euclidean in terms of its "intrinsic" or "internal" geometry.†

These two meanings are linked via the concept of a "point-set." Geometric figures, surfaces and solids alike, are conceived as sets of points. Indeed, entire, unbounded spaces such as $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{C}^1$, etc. are treated as point-sets.

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*[A "surface" is not just a two-dimensional "landscape" embedded in a three-dimensional space. A three-dimensional solid may appear as a "mere" surface in spaces of four or more dimensions.]

But point-sets and other (sub-)*spaces* are sets with "*shape*," with intrinsic organic structure. The "mere collection" concept of sets makes sets out to be quite "shapeless" entities. It holds "shape" (or overall structure) to be irrelevant or extrinsic to the specification of a set. A mere collection is a jumble. Every possible ordering of the same elements constitutes the same set ($\{a,b,c,d\} = \{d,c,b,a\}$).

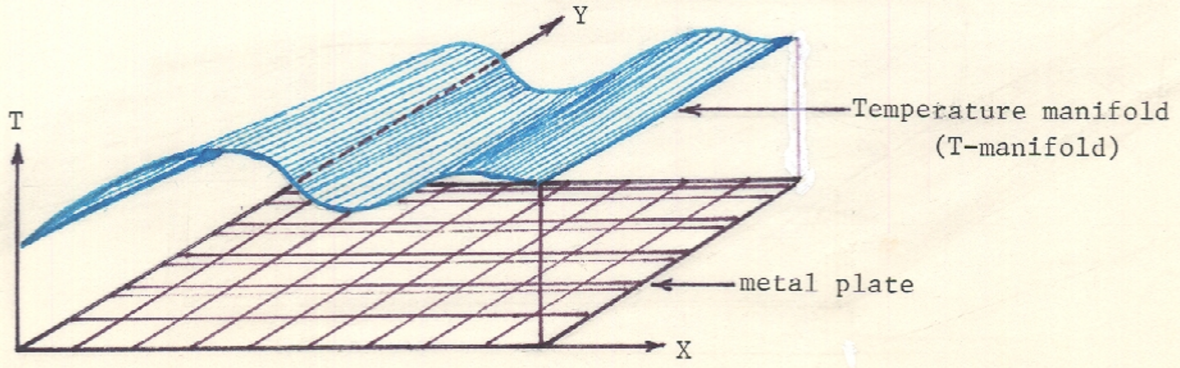
Geometric point-sets, however, have *literal* shape as their essential quality. The *morphology* of a point-set or locus is a definitive property of the set *as a whole*, not reducible to or discernible in the qualities of its "elements" taken separately (these elements being shapeless, featureless points, indistinguishable except for their coordinates; denoting relative position as their only content.) Nor is the "shape" property a function of the cardinal *number* of points contained in the point-set, but only of their *relative* location in the sense of their *internal*, mutual positional inter-relations. The same transfinite cardinal numbers of points are involved in endlessly differing geometric figures. A geometrical figure can be "translated" to a new position in space, thus changing the coordinates of all the points it contains -- i.e. changing the identities of all its elements -- without disturbing its shape-quality. The shape-quality emerges only in "uncountably infinite" aggregates of points, and disappears at the level of points as individual objects. Morphology or intrinsic geometry is thus a truly holistic quality of point-sets.

Geometrical objects, (hyper)surfaces, (hyper)solids, are themselves "spaces." They are *sub*spaces of that space-concept -- typically $\mathbb{R}^n$ -- within which they are conceived. Each such subspace has its own intrinsic geometry, just as entire number-spaces, such as $\mathbb{R}^n$, can embody definite geometries (in the cases of $\mathbb{R}^2$ and $\mathbb{R}^3$, classical Euclidean geometries).

This conceptual convergence of "manifold," "point-set"; "space," and "geometry" is a key to the new operatorial conception denoted above by P.

$\tilde{S}(t) \cdot P^{f(t)}$ signifies one "geometry" ($\tilde{S}(t)$) operating upon, and thereby transforming another ($P^{f(t)}$), one "manifold" remolding another, one "set" multiplying another, one "space" interacting with another. $P^{f(t)}$ signifies whole geometries operating upon themselves, whole manifolds unfolding themselves, sets raised to higher degrees, the self-development of spaces. As might be expected, however, neither the standard "Cartesian" set-product nor the simple Heraclitean product is sufficient for the definition of the rules of point-set multiplication -- of point-sets as operators -- needed here.

Crucial to the usefulness of such a calculus of manifolds is the manner in which manifolds are made to map the states of systems (such as a plasma or an economy). For example, the temperature-state of a square metal plate M can be "measured" by a manifold. Try this thought-experiment. Make two abutting edges of the plate coordinate-axes for the "x" and "y" dimensions of physical space. Erect a vertical axis from their vertex, scaled to Temperature. Above each point of the plate, suspend a "point" of some fabric at such a height, with respect to the Temperature scale, that it represents the temperature measured at that moment "at" the plate-point directly below. Repeat this for the same moment for the rest of the points of the metal plate. The resulting fabric, suspended in three-dimensional x vs. y vs. T space above the metal plate, is a manifold-model of its temperature-aspect, a two-dimensional "temperature-manifold" depicting the temperature-state of the plate not as a gloss-over average, but in all its "geo-graphic" detail.



[Figure 15: 2-d manifold describing the temperature distribution across a metal plate]

The manifold above, the "set of all points" (x,y,T) for the plate, models only a single moment of the temperature-distribution across M. All the measurements for all the points were taken at the same instant. Take all the measurements again at successive instants and, if heat is diffusing through the plate, you get a different "geometry" or "topography" of the T-manifold for each one. You get a succession of point-sets, one for each time t at which you measured T for each point x,y. You get a "set-valued" or "manifold-valued" function of time {all points [x,y, T(x,y,t)]}. To depict this function, we start the manifold moving -- waving like a flag in the wind. I.E., as the time t advances, the temperature-surface will heave and billow, bulge and roll, rise and fall like a sea. The diffusion of heat across the plate will mirror in the T-manifold as waves of temperature-rises rippling across its surface.

Similarly, the temperature distribution throughout a plasma can be represented by a time-varying "topography" or manifold. Only the plasma, a 3-dimensional body, requires a 4-dimensional representation space. Three axes -- x, y, and z -- supply "geographical" coordinates. The fourth, T axis tells temperature. The time-varying manifold {all points [x,y,z, T(x,y,z,t)]} is a changing 3-dimensional "surface" embedded in that 4-dimensional representation space. Likewise, other aspects of a plasma's state -- density-distribution, composition-distribution, field-intensity distribution or charge-distribution, etc. -- can be mapped by the manifold method.

For an economy-model, e.g. for our hypothesis, we would want a productivity-manifold, an indebtedness-manifold, a fictitious-value manifold, etc. In that case the "base-space" -- the variables x, y, z,... -- might not be geographical (physical-spatial) variables, but, for example, various rating-scales for business enterprises. Then the "points" in the base-space would not be "dense," nor would they represent geographical locations. Each point would represent a firm, and the (discontinuous) base-space would be a "taxonomy-space," with a (discontinuous) productivity- or indebtedness- or price- or profitability- etc. manifold erected "above" it. That is, the base-space would be a (non-geographical) classification of the subsystems (e.g. firms) of the system being modeled (e.g. the economy). 103

The manifold method describes the states of the parts (subsystems) and the state of the whole simultaneously, in a single geometrical symbol. The value of the manifold variable (e.g. the temperature value, etc.) at each base-point [e.g. (x,y,3,...,)] describes the state of the subsystem represented by that base-point. The overall morphology or "geometry" of the whole manifold captures the state of the system as a whole. (The unity of this unique "internal geometry" of the manifold can be expressed formulaic ly by the Rienmannian "metric" for the manifold.)

The "manifolds as operators" approach aims to compute future manifold-states by "fleeting" and "reflecting" present ones; by inter"multiplying" and self-"multiplying" them. The time-succession of state-describing manifolds, depictable as the "rippling" motion of the manifolds as time advances, is calculated by the interaction and self-inter-action of the momentary morphology-states of the manifolds earlier in that succession. These successions of manifold-states may be regarded as "manifold-valued functions of time," where time is the traditional abstract, Newtonian "clock time," the "directed" Real variable t, the universal, independent variable, which varies uniformly, independent of the pace and content of the processes occurring in the world. Or, time may be regarded as the product of the action and self-action of the manifolds; as generated by their succession; as nothing but that succession itself.

The type of manifold just described is the type of solution-space defined by partial differential equations, the type of "solution-geometry" corresponding to the solution-functions of such equations. We will call this type the "PD-manifold." There is another kind of manifold, involved in an operator like P in (I) above, which models at a different level of process-description. It corresponds to the solution-functions of "ordinary" or "total" differential equations. We name it the "OD-manifold." Another name is "state-space," Yes, we can treat a whole state-space

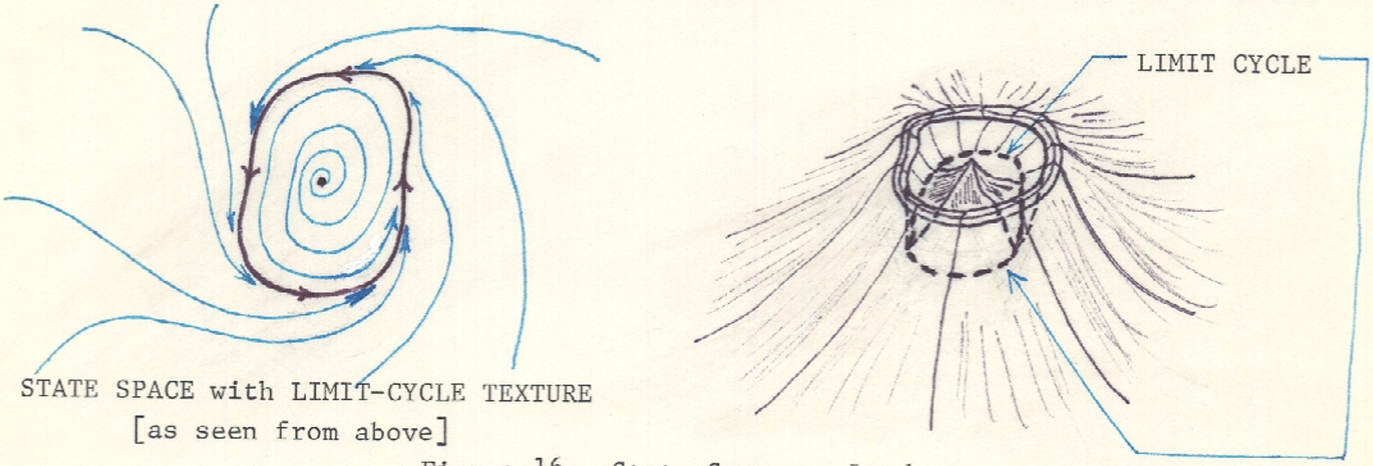
-- the set of all possible states of a given kind of system -- as a manifold. A state-space is a point-set. Moreover, a state-space is a "geometry" or morphology, in a much stronger sense than may at first appear.

Throughout the discussions of limit-cycles, etc., we focused on single trajectories *within* a given state-space. We watched the life-courses of individual systems belonging to a state-space; systems individualized by their different initial conditions or "conditions of birth," symbolized by the different starting-points of their trajectories. Now we treat whole state spaces as single entities. And we want to treat them as *operators*. What would it mean for a state-space to operate on itself? What would such a "product" produce? A reflexive function like $X(t) = P^{f(t)}$ where P denotes a state-space manifold, represents a "self-production," the self-unfolding or self-development of the manifold P. But, if the manifold is a state-space, what is it that is evolving? What is being "self-produced"? What does a whole state-space, as opposed to a single trajectory within it, stand for?

A PD-manifold represents a (physical-)spatial or "geographic" distribution of state-variable values. It shows the variation of a state-measurement across the physical-space body of a system, and thus the state-values for all of its physical parts (points) -- or else likewise for a nonphysical, nongeographical "taxonomy-space" classification of its parts or subsystems. The key to the different meaning of an OD-manifold is regarding it as a *manifold of trajectories*. The set of all trajectories of a state-space expresses the *totality of possible behaviors* for that space.

A state space is the "picture" or geometric counterpart of an ordinary differential equation. That equation expresses the dynamics, the dynamical law, the *law of motion* of a given system in terms of the "movements" or temporal pattern of variation of some measurements on that system, the state-variables. This pattern of variation is depicted as a trajectory. But there are more stable "textural" or "structural" features to a state-space. Some of these features, the "attractors," attract trajectories. Others ("repellors") repel them. Others are neutral ("neutrally stable"). These features include fixed points, limit-cycles, and "strange attractors." But, though they interact with trajectories, they are not "made of" trajectories [for instance, trajectories never touch a limit-cycle attractor ("until" $t = \infty$), unless born there]. It is extremely fruitful to ponder just what these features are made of; just what they are. What makes certain point-sets like limit-cycles and fixed points "stand out" against the rest of the state-space?

Consider a plane state-space dominated by an attracting limit-cycle. The traditional Cartesian axes framing and "coordinating" this space suggest a flat, uniform, Euclidean intrinsic geometry. But the axes are arbitrary. The real essence of this state-space is the "texture" or "structure" of the space as defined by the limit-cycle, and the "pattern" or general "flow" of trajectories in "response" to it. This texture is the state-space geometric symbol of what the differential equation(s) defining the space say(s) formulaically. Therefore, let the axes fade out. Consider the state-plane as a terrain, a landscape you have been (unknowingly) viewing from above, its "texture" the topographic relief lines of that landscape:



STATE SPACE with LIMIT-CYCLE TEXTURE
[as seen from above]

Figure 16: State-Space as Landscape

The limit-cycle is the groove surrounding the central peak at the bottom of a crater. Assuming "gravity," the "path" of any "rock" that falls off the sharp rim of the crater, or off the sharp tip of the peak, is to roll around in toward the "groove" of the limit cycle. This landscape expresses the state-space as a manifold, a geometry, a "multi-curvacious surface" in the sense we need. But what does this geometry mean? What does an OD-manifold model?

It models precisely the same thing modeled by the ordinary differential equation it pictorializes. It symbolizes the dynamical law, the law of change, of the entire range of systems obeying that OD equation. It symbolizes "the laws of Nature" as they apply to those systems.

What does it mean, then, when an OD-manifold in an operator-like P operates upon itself? The self-operation of a PD-manifold just means the carrying out of the temporal process-implications, the following-through on the process-inertia, of the present momentary state of the manifold, toward its next (future) state. It just carries out the given dynamical laws. A state-space landscape, operating on itself, changing its intrinsic geometry, signifies a much more profound kind of change. It signifies change "of higher degree," change of the laws of change, change change, "change squared." It signifies a self-evolution of the laws of Nature themselves; *change of Laws*.

Let us clarify with examples from natural history. Such a "change of geometry," signifying a change of Laws, would describe the transition from molecular to cellular evolution, or the transition within biospheric evolution from a fermentative (heterotrophic) to a photosynthesizing (autotrophic) ecology. In economic history too, it applies at many different levels. On one level, it describes "epochal transitions" like those from the tribal order to "hydraulic"[104] society in the ancient Orient, from the Roman Empire to Medieval society and from Medieval to modern industrial society in Europe. On a different scale, it might apply to the transition *within* modern industrial society between the *state of economic lawfulness* prevailing prior to the Great Depression to that which emerged after World War II.

The kind of profoundly "qualitative" change described by reflexions of OD-manifolds would, if translated back into PD-space, manifest as "singularities," *infinite* quantitative changes, and *discontinuous* change: "shock waves", "tidal waves", and "ruptures" along the manifold.

Observe what this means. Unlike the reflexions among PD-manifolds, an OD reflexion, a state-space reflexion, describes a break in the continuity of previous processes, critical change -- *crisis*. We thus have in sight the mathematical machinery for modeling crisis.

A classical example of "change of geometry" in state-space is the "Hopf bifurcation."[105] Varying the coefficient of a certain nonlinear differential equation induces, at a critical value of that coefficient, a sudden transformation of the texture of its state-space. Initially, the space is dominated by an attracting fixed point. At the critical value of the coefficient, that attractor turns into a repellor, and a bi-stable limit-cycle forms around it.

If the differential equation were such that the critical coefficient itself was a (known) function of time, its value ascending as time advanced until it crossed the critical value, then that equation would describe the *state-space itself evolving*. It would describe the laws of motion evolving, the OD manifold as time-varying, as a function of time.

What we are proposing has deep roots in the Riemannian revolution in geometry. It is strongly analogous to what Einstein's General Relativity does with Riemannian geometry in describing the "gravitational field" concept.

We treat the manifold of trajectories implied by an ordinary differential equation-model as an (in general) non-Euclidean space, even though the state-space from which we start, as defined by its axes, is Euclidean. By observing the general flow of trajectories in this state-space, we are able to determine the "topography" of this "landscape."*

There is, in modern mathematics, a theory of "flows on a manifold" which does not assume, even at the outset, a Euclidean state-space with rectangular coordinates;

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[*] [The "landscape" concept here is a case, not restricted to *biological* applications, of Waddington's "epigenetic landscape."[106]]

which treats state-spaces which are non-Euclidean manifolds to begin with.[107] But let us press on with our considerations on the kind of state-space that has been with us throughout this section, and see where we are led.

State-space is a geometric language, a method of representation of external phenomena, a geometrical symbolization of external processes. It describes the changing measurements on an evolving system by a moving point, tracing a trajectory in a space which is the coordination of those measurements. But at this stage of analysis, we are concerned with a representation *of* this representation. We are viewing state-space trajectories, and whole manifolds of such trajectories, as themselves processes, as *phenomena* to be modeled. We want to model -- conceptually and mathematically -- the mathematical, mental, calculative phenomena of state-spaces. That is, we build a "model of a model," a "model of higher degree"; a "model squared."

Two approaches to describing state-space phenomena suggest themselves, given our heritage of physical theory. One is the Newtonian approach. We describe state-space structures -- repellors, attractors, etc. -- as if exerting "action-at-a-distance" on the trajectory-tracing, system-state-representing points, treated as particles. We postulate "forces" in state-space, conceived as invisible attractive or repulsive influences emanating from the limit-cycle, strange-attractor, fixed-point, etc. structures, and exerting "accelerations" upon the system-points, pulling their trajectories in, or pushing them away. These are conceived, in analogy with the physical-space force concepts, as "state-space gravity" or "state-space electromagnetism," etc. We would try to write equations relating these forces to velocities and accelerations along the state-space trajectories. Using mathematical models, analogous to Newton's F = Ma, aiming not only to describe but *predict* trajectories, given certain initial data, we would formulate a "mechanics" of state-space.

However, conceiving these state-spatial "forces" in terms of force-*fields* would start us on a different track. Fields might be conceived, at first, merely as organisms of "lines of force" -- "lines" in no way part of, but merely imbedded in, a state space still conceived along the lines of the Newtonian, inert, passive "Absolute Space" concept of physical space. But working on with this concept would tend to move it in the direction of a second, Einsteinian-Riemannian approach.

In General Relativity theory, space *is* the field; the field *is* space. The field is the structure, the geometry of space. Bodies of mass must obey the geometry of physical space. They are forced to follow its contours in their motion. These contours are not everywhere rectilinear or "flat." Physical space is a non-Euclidean manifold. The curvature, the "fold-structure" or landscape of this manifold *is* the "gravitational field." The degree of curvature, the extent of deviation from flatness or linearity in a given locality, expresses the strength of the gravitational field at that locality (just as the *steepness* of our state-space landscape expresses the intensity of attraction or repulsion around an "attractor" or "repellor"). Particles of mass traveling along the contours of space, as they must, give the appearance of "gravity" -- of an invisible attracting force "acting at a distance."

In this theory, physical space is no longer Aristotelian-Newtonian space, absolute, passive, and inert. Space is no longer a mere "theatre" or "container" for matter-energy and its "events." Space is an active co-participant in process. The geometry of space becomes a dynamical variable, a function of time -- that is, a function of the processes and events taking place "in" it. Space changes. The geometry, the space *evolves* along with its "contents." Indeed, space and matter *interact*. Not only does the geometry of space tell masses how to move. The presence and motion of matter in turn *shapes space*, tells space how to bend. This is the Riemannian revolution. Geometry is not fixed, static, "nounic" -- outside of process. Space is not exempt from evolution. Space, the forming, molding background and vessel of universal evolution, is also part of the evolution, part of the contents -- part of "matter" in that sense. Or, matter is just concentrated field -- "dense space." In fact, the ultimate tendency of Einsteinian theory, as of Riemannian theory before it,[108] is a "Unitary Field Theory" in which mass-energy-space-time is conceived as a single medium, a single self-differentiating, self-evolving fabric.

When we talk of "manifolds as operators", "point-sets as operators", "whole geometries as operators", "landscapes as operators", we mean _fields as operators_. We are working toward a "calculus of fields." We want an "arithmetic" and "algebra" capable of symbolizing entire fields with single symbols, and of _computing predictively_ with such symbols, calculating the interactions and self-interactions of generalized fields; treating field-symbols as, in effect, generalized "_numbers_."

In this Riemannian context, our modeling programme can be seen as tending to a generalization of General Relativity in two senses.

(1) In General Relativity, only the grossest feature of matter in motion -- its mass -- operates upon the geometry of space-time. This is what confines its descriptive powers to "physics" -- i.e. to the lower stages of cosmic evolution and to the gross features of contemporary planetary and stellar locomotion. General Relativity can't see past the atomic level of matter. It grosses over the internal structure of a body of mass. It is insensitive to distinctions of the degree of organization of matter. Consequently, it can't "see" evolution. But certainly the organization of a piece of matter, such as the planet Earth, affects the subsequent evolutionary dynamics in its vicinity just as its mass affects the subsequent gravitational dynamics. Mass affects the spatial motion potentialities, and the level of organization affects the potentialities for further evolution.

We want a "geometrodynamic" theory capable of describing social, economic evolution (and, by implication, metabiotic, cellular, molecular, and atomic evolution before it). Within social evolution, we want a theory capable of modeling economic growth and crisis.

In other words, we want to model kinds of "motion" -- kinds of _change_ -- more general than mechanical motion, mere change of position in physical space. We want to model growth and evolution, including "novelty," the creation of new qualities in the universe, as kinds of motion; to generalize the concept of "motion" to encompass evolution.

The language that depicts change of organization, change of state, as change of position in a space, is that of state space.

Other state-variables -- other than mass -- must operate upon the space-time manifold. Mass affects principally its "physical-space" geometry. Other state-variables -- state-variables reflecting the state of organization of bodies -- can be modeled as affecting the state-space geometry, as "bending" the state-space continuum, altering the landscape, changing the state-space "field." The traditional "Euclidean" state-space for autonomous dynamical systems, employed here throughout, already embodies this principle in a rudimentary way. For different individual systems of the same species, belonging to the same state space, differently-shaped, differently "bent" trajectories flow from different initial conditions, different initial values of the state-variables. But the minimally adequate arena for the representation of this "Unified Field" model of universal evolution* is "state/space-time." That is a generalized state-space/phase-space which incorporates the "physical" space-time manifold -- in the form of the "geographical" positional variables $x(t)$, $y(t)$, $z(t)$, plus the clock-time variable t, taken as state-variables -- along with the more traditional kinds of state-variables.

In General Relativity, mass is mass-energy. That is the meaning of $E = MC^2$. But this "physical" energy concept is inadequate to capture the qualitatively higher energizing principles of biological and social "matter." "Matter" evolves. So does "Energy." We need a concept of energy which can accomodate such an evolutionary hierarchy of energies, as well as one which measures the organization of matter as an aspect of its (mass-)energy level. An already available concept which fuses considerations of "energy" and "order" is that of "negentropy." This concept measures not just the "cardinality" of material bodies, counting the *number* of particles (atoms) they contain ($\propto$ mass), but the quality of *interconnection* of these components, their internal structuration as well.[109] The approach to generalizing General Relativity would then be as follows: the local "bending" of the state/space-time manifold is proportional to the local concentration of negentropy. The bending of physical/space-time in proportion to mass-energy concentration is retained as a special case.

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*[The "Unified Field Theory" modeling programme described here involves primarily a "linguistic" unification in terms of the mathematical language. This does not, in and of itself, resolve the substantive problems of classical "Unitary Field Theory," though it may contribute insight toward such resolution.]

Riemann found that the intrinsic geometry or "topography" of a manifold or "landscape" could be captured by a function, called its "metric." The metric is the manifold's internal "distance function." It measures the distance between points on the manifold, in terms of paths between them *confined to* the surface of the manifold. The curvature or "many-fold-ness" of the manifold or hypersurface is what makes its internal distance relations different from those of a flat Euclidean plane or hyperplane. Therefore the metric captures the "general curvature," the essence of the particular geometry of a given manifold.

Of course, the "many-er" the "folds" of the manifold, the more complex the topography of its landscape--the more variable and multiform its curvature--the more difficult it is to determine an overall, unified formula for the metric which is *invariant* for every locality on the manifold. The unifying principle of the manifold's morphology may elude formulation.

But, however difficult it may be to express the metric formulaically, its importance for characterizing a state space, viewed as a manifold, is manifest. The metric measures distances between points. But points, in state-space, represent states. The metric of a state space "landscape" or "field" tells us how far apart one state is from another. It tells the lengths to which a system must go to get from one particular state to another. It describes the mutual "accessibility relations" among states, their proximity and remoteness relations.

Earlier we characterized state-spaces or OD-manifolds as geometrical symbols of the restrictions on state-transitions, the constraints on possible trajectories, the behavioral limits, i.e. the laws of motion of the systems they model. The metric retranslates this geometric symbol of natural law into an "algebraic" (formulaic) one, and into one of a form much less opaque than that of the original differential equation model.

A promising approach to advancing this programme involves linking the geometric concept with the negentropy concept. The negentropic state of evolving mass-energy at each moment would be expressed by the geometry of the unitary-field (world-manifold) at that moment. The metric of that manifold would serve to measure the negentropic value at each locality.

(2) In General Relativity, the laws of Nature are still conceived as fixed. The manifold of physical space-time changes its local geometry in response to the local doings -- densations and rarefactions -- of mass-energy. But the laws of these doings and of this response do not change. Evolution leaves them untouched, never reaching their depth.

In reflexive functions like (I) above, the whole state-space manifold will, after some duration, complete a full self-operation. This self-reflexion will alter the global geometry of the state-space. This signifies a change in the "state of lawfulness" or "laws of Nature" which the state-space "field" represents. Thus, in this model, the laws of Nature belonging to a given stage of natural evolution change gradually, until they come to a crisis-point from which they emerge transformed. The laws themselves evolve, are subject to evolution, along with the (rest of the) "content" of "Nature." The universe, the unitary field, propagates its evolution as a succession of qualitatively different universes.

Such a mathematical metaphor provides a powerful linguistic tool for predictively and retrodictively* modeling the known pattern of cosmic evolution. For instance the atoms-only* universe of stellar-atomic evolution is, in terms of its dynamical, behavioral potentialities, a different universe from the universe which comes into being with the advent and proliferation of the molecular organization of matter, molecular evolution. Likewise, the fermentative biosphere differs sharply, "discontinuously" from the photosynthetic one which succeeds it and subsumes it. Similarly, feudal-agrarianism presents a flavor of society contrasting sharply with that of the capital-based industrial society which succeeded it, and with that of the tribal-imperial chattelist society which preceded it.

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*[The use of mathematical models to reconstruct the past as well as to forecast the future is a neglected area of applied mathematics, and awaits more powerful modeling methods. However, some "retrodictive" work has already commenced.[110]]

To model such crisis-mediated transitions, evidently, the flexion and reflexion of whole state-spaces, whole OD fields, is requisite.

Applying the "fields as operators" conception to economy-modeling yields an expected solution-function of the same general form as that for the plasma model (I) above:

$$\begin{aligned} S(t) &= g(t)[E(t)]^{f(t)} = [\tilde{S}(t)][E(t)]^{f(t)} \\ &= [\tilde{E}(t)]^{h(t)} \cdot [m(t)] \cdot [E(t)]^{f(t)} \qquad \text{(II)} \end{aligned}$$

We have modified (II) vis-à-vis (I) in two ways. First, we have made explicit the "internal degrees of freedom" of the "economy operator" by making it a function of time, $E(t)$ (see the observations on $\mathbb{H}$, part 5: "Hyperfunction Modeling of Creative Process," pp. 121-2 above). Second, we have replaced $\tilde{S}(t)$ in the final term with the dynamical reflexion of the complement of $E(t)$, flecting a mediating factor, $m(t)$. That factor represents the "boundary" or "borderline" systems, belonging to both (or neither) of the complementary operators $E(t)$ and $\tilde{E}(t)$, which are involved in mediating the interaction between those operators.

As outlined earlier (see pp. 191-2 above), the self-operation of $E(t)$ involves the determination of the successive geometry-states of a number of PD manifolds, including a productivity-landscape or "field," a profit-field, wages-field, and various price-fields, as well as an indebtedness landscape, an employment landscape, a fictitious value landscape, and others. These may have either "geographical" or "categorical" base-spaces (e.g. "taxonomy-spaces"). Some of these are indeterminate or intergenerate -- i.e., calculable by flexions of others. All of these represent the detailed state, or aspects of the state, of the world economy. More directly, E(t) would involve the OD-manifold for the economy as a dynamical system. The OD-manifold(s) in E(t) move cumulatively, but much more slowly than the PD-manifolds, toward full self-reflexion.* For a full OD self-reflexion may mean, as we have seen, a crisis and critical transition into a new evolutionary stage of economy, in which the old PD-manifolds may lose their former forms and meanings.

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*[atoms, or subatomic entities only]

Successfully representing the "economy operator" (economy "verb"), E(t), and the economy's state, S(t), via an "array of manifolds" or "operator-fields" in this way would provide a dynamic model which is both detailed (highly disaggregated in effect) and holistic (macroanalytical). The obsolescence depreciation effect, the self-devaluation (or self-inflation†) of capital, as well as the other moments of our interconnection hypothesis, would be integrated features of this more general model, within which their relative impact and importance could be assessed.

The implementation of this "manifolds as operators" modeling programme is a formidable task of linguistic design. Though instructive, neither traditional set operations such as the Cartesian product, nor our "Heraclitean product," nor even Tensor operations appear to be adequate to the "Field Product" rules we must discover. Moreover, the equations to which reflexive "field-functions" like (I) and (II) above could be solutions are evidently of a kind more general than anything encompassed by either ordinary or partial integrodifferential equations, or even by Tensorial equations.

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*[If PD-manifolds are included directly in the "array of geometries," E(t), then the delay in OD reflexion vis-à-vis PD reflexion can be arranged via a fractional exponent, constant or time-varying, on the OD-manifold operator, making its degree always a fraction of the value of f(t).]

†[See Section I, pp. 16-22]

The core-operator "arrays" such as P and E(t) must be so designable that only flexions (interactions) occur where that is appropriate, but so that self-flexions or reflexions occur where that is called for, and so that OD- and PD-manifolds are properly related and translated, when necessary, one into the other. If we want a continuous model, the exponent of the core operator must be a continuous function such as f(t). This requires that the fractional and irrational powers of the core operators and geometries must be defined and represent the correct intermediate states of plasma, economy, or whatever eventity is being modeled.

I suspect that a hypernumber formulation of these geometries is the key to successful implementation. In ⋕, each point is also, explicitly, an operator; stands for a definite operation. Manifolds of ⋕-points from different regions (subspaces) of ⋕ would represent different operational characteristics and therefore, different rules of "manifold multiplication," even if they shared an identical topography or intrinsic geometry. Thus the same manifold in a different systemic context could have very different flexive and reflexive effects.

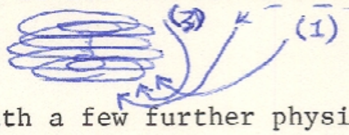
The Pervasiveness of Self-Reflexive Process

The impetus and insight which led to the simple "hyperfunction" model-form was provided by the reflexivity paradigm. That paradigm, in conjunction with the hyperfunction form, then indicated the new kind of mathematical object -- the manifold-as-operator -- necessary for the hyperfunction form to achieve a fuller descriptive power, with great gains in notational simplicity and clarity as by-products. That paradigm is ideographically expressed by the resulting general hyperfunction form:

$$S(\overset{\boxed{+\Delta t}}{t}) = [\tilde{S}(t)][F(\overset{\text{or } -\Delta t}{t})]^{f(t)} , \quad \leftarrow \text{errors} \quad \otimes \qquad \text{(III)}$$

wherein $S(\overset{+\Delta t}{t})$, $\tilde{S}(t)$ and $F(t)$ denote "fields-as-operators."

The very structure of (III), given the "eventity" operator concept developed throughout this section, expresses, almost mimetically, the structure of process. (III) states that the state of a system is the product of its environment's impact with its self-impact. A system at any moment is the joint result of the effects of its environment's current (and) past activity upon it and the effects of its own past activity upon it.



(2) ≡ The (state of the) input medium itself

Let us end these considerations with a few further physical examples which fall under that concept which has been their main result, namely the reflexivity paradigm. This will serve to solidify the sense of the concreteness of its meaning, and of the pervasiveness of its application to natural phenomena, before we plunge into exclusive concentration upon its economic modeling ramifications.

Our first example spans the disciplines of subatomic physics, chemistry, and biology. Our second example goes right to the heart of our earlier discussion on General Relativity.

In population biology there is a famous nonlinear differential equation model of predator-prey population dynamics known as the Lotka-Volterra-Verhulst model.[111] For a one-predator species, one-prey species system, with the $X = x(t)$ the state-variable giving prey population as a function of time, and with $Y = y(t)$ the predator population function, its simplest form is:

$$(1) \quad DX = aX - bXY - cX^2 \qquad \text{(a through k are \underline{constants})}$$

$$(2) \quad DY = -hY + kXY$$

It says that the rate of change of the prey population is proportional to its present population (aX^1) minus the flexively nonlinear term for the interaction of prey and predator (bXY), minus a reflexively-nonlinear (quadratic) term in the prey-function itself (cX^2). The "interaction" term bXY describes the eating of prey by predators, and thus clearly subtracts from the rate of growth of prey, DX. But what of cX^2? This term is described in the literature as the "self-interaction term."[112] It describes the collective self-interaction of the prey population, the "intra-action" of that population as a whole (not, directly, the self-interactions of individual members of the prey population).* The term cX^2 describes the self-limitation of the animal prey species' population even in the absence of predation.

The modeling efficacy of this equation system is not limited to biological applications, however. Models of the same ideographic form, though of course with different meanings assigned to the state-variable functions $x(t)$ and $y(t)$, describe the "photon autocatalysis" of the quantum-physical laser process of "nonlinear optics." In chemistry: "the nonlinearity in chemical kinetics, for instance, may arise through autocatalysis and through positive feedback mechanisms."[113] Autocatalysis or "self-catalysis" is the process wherein molecules of one chemical species in a solution react with their chemical environment so as to produce more of their own chemical species. Auto-inhibition is the reverse process. Again, equation-models of the same symbolical form as the Lotka-Volterra-Verhulst system are used to describe this process. And in this case too, the "self-interaction" described is not directly at the "part" or "each molecule with itself" level, but at the level of the (more or less organized) population of the self-increasing or autoinhibiting chemical species as an active agency within the chemical medium.

Our second example pertains to the meaning of the nonlinearity of Einstein's gravity field equations for General Relativity. One physicist describes the "autocatalytic" self-interaction of the gravity field in this way:

> Einstein's...Relativity deals with the nonlinearities that are implicit in the gravity field. Gravity is a form of energy, energy has mass, mass creates gravity, so therefore gravity makes more gravity.[114]

Another physicist puts the matter as follows:

> The self-interaction of gravitation...is one manifestation of the fact that gravity acts on "everything"...We then have a self-interacting gravitational field satisfying a nonlinear field law.[115]

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*[That population should indeed be regarded as forming a whole, and not as a mere linear "sum" of biological individuals.]

In terms of our modeling language, this "self-interaction" of gravity would be modeled as the self-reflexion of a PD-manifold, not of an OD-manifold. This self-interaction involves the intensity landscape of the gravitational field across physical space(-time), and not any alteration of the fundamental law of gravity, which is just what the field equation itself formulates.

The prevailing theoretical disposition is such that explicit discussion of examples of self-reflexive process is ordinarily encountered only in the form of extreme cases on the outer frontiers of logic, mathematics, physics, etc., where it typically arises as a matter of seemingly intractable or near-intractable difficulty. This theoretical disposition involves the unconscious and linguistic-grammatical roots we have discussed.* In the foregoing, we have presented evidence that a changed disposition with regard to self-reflexive phenomena can lead to very different results. A changed disposition in this regard may also reveal reflexive process to be pervasive in even well-studied areas where the prevalence of linear paradigms has made it difficult to discern and easy to overlook.

The next step for our modeling programme is to search out such overlooked involvements of self-reflexive process in the realm of economic dynamics, with scrutiny to the relatively neglected dimensions of cyclical dynamics, credit and monetary processes, and technical change, especially to the interconnections of these.

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*[See pages 32-34; 39-43; 48-50].

III. APPENDIX

BASIC PAPERS ON HYPERNUMBERS

1. C. Musès, "Applied Hypernumbers: Computational Concepts," Applied Mathematics and Computation, 3 (1977), pp. 211-226.

2. C. Musès, "Hypernumbers II: Further Concepts and Computational Applications," Applied Mathematics and Computation, 4 (1978), pp. 45-66.

3. C. Musès, "Computing in the Bio-Sciences with Hypernumbers: A Survey," International Journal of Bio-Medical Computing, 10 (1979), pp. 519-525.

4. C. Musès, "Hypernumbers and Quantum Field Theory (with a Summary of Physically Applicable Hypernumber Arithmetics and their Geometries)," Appl. Math. and Comput., 6 (1980), pp. 63-94.

REFERENCE NOTES

1. Hans Bremermann, "Transcomputationality," (unpublished), 1972.

2. Donald Harris, *Capital Accumulation and Income Distribution*, Stanford University Press (Stanford, CA: 1978), p. viii.

3. John von Neumann, "A model of general economic equilibrium," *Rev. Economic Studies* (London, England), vol. 13(1), pp. 1-9; 1945-6.

4. Cited in Norman Zabusky, "Coherent Structures in Fluid Dynamics," *Proc. Orbis Scientia Conf.*, University of Miami (Coral Gables, Fla.: 1977).

5. Ilya Prigogine, "Order Through Fluctuation: Self-Organization and Social System," in Jantsch and Waddington, eds., *Evolution and Consciousness: Human Systems in Transition*, Addison-Wesley (Reading, Mass: 1976), p. 122.

6. "Linearization is essentially the only tool we possess for the study of nonlinear systems."
   Robert Rosen, *Dynamical System Theory in Biology*, vol. I., Wiley Interscience (New York: 1970), p. 166n.

7. "...it has been argued that every nonlinear problem is really individual; that is, it requires individual methods, usually very complicated and difficult methods, and it is rather improbable that one can learn from one nonlinear problem to solve another nonlinear problem ...In fact I do think that all these problems have some features in common..."
   Werner Heisenberg, "Nonlinear Problems in Physics," *Physics Today*, May 1967, p. 27. See also Ladis Kovachs, "Life Can Be So Nonlinear," *American Scientist* 48:2, June 1960, p. 224.

8. W. Ross Ashby, Introduction to Cybernetics, John Wiley and Sons (New York: 1963), p. 37. [quoted p. 50, PO&LC (footnote 38)]

9. "'Chaos' (the strange attractor) is, besides steady state and limit cycle, one of the few basic modes of qualitative behavior possible in nonlinear dynamical systems."
   Otto Rössler, "Different Types of Chaos in Two Simple Differential Equations," *Z. Naturforsch* 31a (1976), p. 1664.

10. "Within the past dozen years a surprising number of 'model' partial differential equations describing the time evolution of numerous mildly nonlinear phenomena have been found to share a growing family of quite remarkable properties, including...Moreover there exist many relations between these various properties, some direct and obvious, others ad hoc and mysterious. One has the feeling that an important, central, underlying and unifying principle remains to be discovered."
    R. Palais, "The Soliton Syndrome," (Handout for "An Elementary Expository Lecture" to the U.C. Mathematics Dept., Summer 1978).

11. Ilya Prigogine, *op. cit.*, p. 95 ff: "...classical thermodynamics permits the interpretation of *equilibrium* structures...Dissipative structures are associated with an *entirely* different ordering principle, which may be called *order through fluctuation*. In fact, such structures arise from the *amplification of fluctuations* resulting from an instability of the 'thermodynamic branch'...In order to be able to take form, dissipative structure requires a nonlinear mechanism to function. It is this mechanism which is responsible for the amplification mechanism of the fluctuation."

12. Joseph Persky and José Alberro, "Technological Innovation and Profit Rates," URPE Conf. (unpublished), p. 2.

13. *Ibid.*, p. 16.

14. *Ibid.*, p. 1-2.

15. Robert Peters, *Return on Investment*, American Management Association Committee on *Management (AMACOM)*, 1974, pp. 1-5; 33; 44.

16. Thorstein Veblen, *The Theory of Business Enterprise*, Charles Scribner's Sons (N.Y.: *1904)*, pp. 229-234.

17. Karl McEachron, "Engineering Education is Closing the Gap," *Case Alumnus*, April 1958, p. 18.

18. Lyn Marcus, *Dialectical Economics*, D.C. Heath & Co. (Lexington, Mass: 1975) pp. 8-10.

19. A.M. Milne and J.C. Laight, *The Economics of Inland Transport*, Sir Isaac Pitman & Sons (London: 1965), pp. 191-192. See also: David R. Anderson and Leo A. Schmidt, *Practical Controllership*, Irwin, 1961 (revised), pp. 488-9; Eugene L. Grant, *Principles of Engineering Economy*, Ronald Press (N.Y.: 1950).

20. David R. Anderson and Leo A. Schmidt, *ibid*.

21. *Ibid.*, pp. 352-353.

22. *Ibid.*

23. A.M. Milne and J.C. Laight, *op. cit.*, pp. 232n.

24. *Ibid.*, pp. 191-2.

25. Heinz von Forster, "Logical Structure of Environment and its Internal Representation", Internat'l Design Conf., Aspen, Colo., *Annual*, 1962, *passim*.

26. cf. Lyn Marcus, *op. cit.*, p. 419.

27. Edwin Mansfield, *Principles of Macroeconomics*, W.W. Norton (N.Y.: 1974), pp. 311-312.

28. Arthur M. Okun, "The Great Stagflation Swamp", The Brookings Bulletin 14:3 (Fall 1977) pp. 1-7.

29. Benjamin Lee Whorf,"Languages and Logic" in Language, Thought and Reality, M.I.T. Press (Cambridge, Mass: 1956), pp. 241-2.

30. Of course, "Pure operation" signs can be reformulated as operators or functions. Thus, typical binary operations * in a*b may be written as *(a,b) wherein * is being treated as a function operating upon a two-component ("1 by 2") "array" or (generalized) "vector" (a,b). The "unmediated product" ab of our "process algebra" concept might, in these terms, be denoted $[f(a,b)](a,b)$--i.e. with $* \equiv [f(a,b)]$--since the operation going on between a and b is a function (f) of both a and b. In "associative" cases, $[f(a,b)](a,b) = f[(a,b)(a,b)] = f(a,b)^2$. Note how this notation clarifies the noncommutativity so common among higher kinds of operators (e.g., hyper-numbers):

$ba = [f(b,a)](b,a) \neq [f(a,b)](a,b) = ab$, because $(a,b) \neq (b,a)$.

See David M. Burton, Abstract and Linear Algebra, Addison-Wesley (Menlo Park: 1972), pp. 40-41. Note f above is explicitly an "operation-valued" function of "ordered pairs" or "couples" (or generalized 2-dimensional row "vectors") (a,b).

31. Piero Sraffa, Production of Commodities by Means of Commodities, Cambridge U. Press (N.Y.: 1960).

32. Bertrand Russell, The Principles of Mathematics, W. W. Norton (N.Y.: 1902), p. 104.

33. G. Spencer Brown, Laws of Form, Bantum (N.Y.: 1973), pp. xviii; 99-100.

34. Bertrand Russell and Alfred North Whitehead, Principia Mathematica, (to * 56), Cambridge U. Press (N.Y.: 1970), p. 61.

35. Ibid., p. 37.

36. Susan K. Langer, An Introduction to Symbolic Logic, Dover (N.Y. : 1953, 2nd Ed.), p. 62.

37. George Boole, An Investivation of the Laws of Thought, Dover (N.Y.: 1958), p. 53.

38. G. Spencer Brown, op. cit., pp. xvii-xviii; 97-100.

39. Ilya Prigogine, Gregoire Nicolis, Agnes Babloyantz, "Thermodynamics of Evolution", Physics Today, Nov. 1972 (pp. 23-28); Dec. 1972 (pp. 38-44).

40. All trajectories are attracted to the fixed point, if the system is a stable one (real part(s) of eigenvalues negative, creating decreasing exponentials). If the system is unstable (real part(s) of eigenvalues positive, creating increasing exponentials), all trajectories are repelled. See Morris W. Hirsch and Stephen Smale, Differential Equations, Dynamical Systems, and Linear Algebra, Academic Press (S.F.: 1974), pp. 89-97.

41. George Boole, *op. cit.*, p. 170.

42. *Ibid.*, p. 101.

43. Nicholas Rescher, *Many Valued Logics*, McGraw-Hill (N.Y.: 1969), pp. 89-90. Rescher notes that paradoxes like the pseudomenon can be solved, in appropriately "arithmetic" ideographies of formal logic, by fractional truth-values such as 1/2. He also notes that this is tantamount to entertaining an *infinivalent* instead of the traditional Aristotelian bivalent logic, i.e., to admitting a *continuum* of truth-values.

44. George Boole, *op. cit.*, p. 49.

45. *Ibid.*, p. 51.

46. George Boole, *The Mathematical Analysis of Logic*: Being An Essay Towards A Calculus of Deductive Reasoning, Barnes & Noble (N.Y.: 1948), p. 17.

47. George Boole, *An Investigation of the Laws of Thought*, *op. cit.*, p. 50n; 51.

48. Charles Musès, "Hypernumbers II: Further Concepts and Computational Applications", *Applied Math. & Comput*. 4: 1978, pp. 45-66 (*passim*).

49. Charles Musès, "Applied Hypernumbers: Computational Concepts", *Applied Math. & Comput*. 3: 1977, p. 213.
See also Charles Musès, "Explorations in Mathematics", *Impact of Science on Society*, special issue on the frontiers of science, *27*:1 (1977), p. 83.

50. George Boole, *The Mathematical Analysis of Logic*, *op. cit.*, p. 70.

51. Charles Musès, "Hypernumbers II", *op. cit.*, p. 46.

52. Charles Musès, "The First Nondistributive Algebra" in E. R. Caianiello, Ed., *Functional Analysis and Optimization*, Academic Press (N.Y.: 1966).

53. Thomas L. Saaty and Joseph Brown, *Nonlinear Mathematics*, McGraw Hill (N.Y.: 1964), p. 1.
See also: Charles Musès, "Aspects of Some Crucial Problems in Biological and Medical Cybernetics", in *Progress in Biocybernetics*, vol. II, Norbert Wiener and J. P. Schadé, Eds., Elsevier (N.Y.: 1965), pp. 215-217.

54. Charles Musès, "Hypernumbers II", *op. cit.*, pp. 61-62.

55. Charles Musès, "Hypernumbers and Quantum Field Theory (with a Summary of Physically Applicable Hypernumber Arithmetics and their Geometries)", *Appl. Math. Comp. 6* (1980), pp. 79-80.

56. Charles Musès, "The First Nondistributive Algebra", *op. cit.*, p. 207.

57. Morris W. Hirsch and Stephen Smale, *op. cit.*, pp. 57-8.

58. *Ibid*., pp. 135; 139; 143.

58.5 Charles Musès, "Hypernumbers II", *op*. *cit*., p. 53; see also "Applied Hypernumbers: Computational Concepts", *op*. *cit*., p. 217.

Note that if we name $A \equiv \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \leftrightarrow (1 + \varepsilon_3)$, then

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} \text{ although } \det(A) = 0;$$

$$A^{0} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \leftrightarrow (1 + \varepsilon_3)^{0} = \frac{(1 + \varepsilon_3)}{2} ;\ A^{-1} \cdot A^{+1} = A^{0}$$

$$= \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = A^{0}; \text{ and } (A^{0})^{n} = \frac{(1 + \varepsilon_3)^{n}}{2^{n}} = A^{0}.$$

59. Charles Musès, "Applied Hypernumbers: Computational Concepts", *op*. *cit*., p. 213; "Explorations in Mathematics", *op*. *cit*., p. 83.

60. Sylvanus P. Thompson, *Calculus Made Easy*, MacMillan Co.; 2nd ed., 22nd printing (N.Y.: 1952), "On True Compound Interest and the Law of Organic Growth", pp. 134-164.

61. Charles Musès, "Systemic Stability and Cybernetic Control: An Introduction to the Cybernetics of Higher Integrated Behavior", in *Cybernetics of Neural Processes*, NATO Internat'l School of Physics, April 26-May 13, 1962, Proc. Ed. by E.R. Caianiella (Rome: 1965), p. 196.

62. Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*, Routledge & Kegan Paul (London: 1974), pp. 17; 42-3.

63. Banesh Hoffman, *About Vectors*, Dover (N.Y.: 1966), pp. 105-6.

64. That the hypernumber differential operator is "with respect to time" does not exclude partial differential equations from hypernumber "hyperalgebraic" treatment. If the PDE solution-function-value involves space dependencies, or if other non-t variables are involved in its argument, these other variables are treated as time-parametrized functions; e.g. P(X,Y,Z,T) = P(x(t), y(t), z(t), t).

65. Charles Musès, "The First Nondistributive Algebra", *op*. *cit*., pp. 182-3.

66. Charles Musès, "Hypernumbers II", *op*. *cit*., pp. 45-6.

67. G.W. Scott Blair, *Measurements of Mind and Matter*, Philosophical Library (N.Y.: 1956), p. 42.

68. (see reference note 65 above).

69. (See p. 106 of text; see reference note 67.)

70. George Boole, *The Mathematical Analysis of Logic*, op. cit., pp. 16-17n.

71. Charles Musès, "The Meaning of Cybernetics in Revolutionary Systems (Those with Rapid or Discontinuous Shifts in Standards)", *Proc. of the Internat'l Conf. on Cybernetics*, J. Rose, Ed., Gordon & Breach (New York: 1970), p. 188.

72. Charles Musès, "Applied Hypernumbers: Computational Concepts", *op. cit.*, p. 217.

73. Charles Musès, "Computing in the Biosciences with Hypernumbers: A Survey", *Internat'l Journal of Bio-Medical Computing 10*: 1979, pp. 521-522. See also "Hypernumbers II", *op. cit.*, pp. 52-54; 62, and "The First Non-Distributive Algebra", *op. cit.*, p. 208.

73.5 Dr. Musès claims that only these operational geometries, or power-orbit configurations, correspond to "viable" (non-self-contradictory) arithmetics and algebras. See "Hypernumbers II", *op. cit.*, pp. 63-4.

74. See for example Francois Meyer, "L'Accélération de L'Évolution", Chapitre II in *L'Encyclopédie Française* 20: pp. 20.24-1 to 20.24-6 (Paris: Larousse, 1959). L. S. B. Leakey, "I: The Progress of Man in Africa" in *The Progress and Evolution of Man in Africa*, Oxford U. Press (London: 1961), pp. 22-3: "...There seems to me to be an inexorable natural law concerning the speed of progress...Progress...seems to proceed faster and faster and faster in what might perhaps be termed inverted geometrical progression so far as the time factor is concerned. This seems to be a law of Nature."

75. Whorf provides an illuminating discussion of this crucial distinction of *force* from mere momentum, in one of his essays on the unconscious ontological and epistemological conditioning involved in the habitual practice of a given natural language from early childhood: "...This notion of storing up power, which seems implied by much Hopi behavior, has an analog in physics: acceleration. It might be said that the linguistic background of Hopi thought equips it to recognize naturally that force manifests not as (mere) motion or velocity, but as cumulation or acceleration. Our linguistic background tends to hinder us in this same recognition, for having legitimately conceived force to be that which produces change, we think of change by our linguistic metaphorical analog, motion, instead of by a pure motionless changingness concept, i.e. accumulation or acceleration. Hence it comes to our

naïve feeling as a shock to find from physical experiments that it is not possible to define force by motion, that motion and speed, as also "being at rest", are wholly relative, and that force can be measured only by acceleration."

Benjamin Lee Whorf, "The Relation of Habitual Thought and Behavior to Language", in *Language, Thought and Reality*, *op. cit.*, p. 151n. Evolutionary acceleration, not spatially oriented like the force and acceleration vectors familiar from mechanics, might be formulatable as a "time-like" vector -- a "vector" directed along the time dimension, or "in the direction" of time.

76. However, see Florian Cajori, *A History of Mathematical Notations*, Vol. I, Open Court Publishing (La Salle, Ill.: 1974), §313, pp. 358-9.

77. cf. G. Spencer Brown, *Laws of Form*, *op. cit.*, pp. viii-xi. See also: (1) Gotthard Günther, "Time, Timeless Logic, and Self-Referential Systems", *Annals N.Y. Acad. Sci.*, Vol. 138, Article 2, February 6, 1967, pp. 396-406; (2) "Formal Logic, Totality, and the Superadditive Principle", Biological Computer Laboratory, University of Illinois, Urbana, Report No. 3.3; (3) "Cybernetic Ontology and Transjunctional Operations" in M. C. Yovits, G. T. Jacobi, and G. D. Goldstein, Eds., *Self-Organizing Systems*, Spartan Books (Washington, D.C.: 1962), pp. 313-392; (4) "Die philosophische Idee einer Nicht-Aristotelischen Logik", *Proc.* XIth Internat'l Congress of Philosophy, Brussels, Aug. 20-26, 1953, *5*: 44-50.

78. George Boole, *An Investigation of the Laws of Thought*, *op. cit.*, p. 50n.

79. See Ernest Nagel and James R. Newman, *Gödel's Proof*, New York University Press, (N.Y.: 1958), pp. 68-84.

80. See Nicholas Rescher and Alastair Urquhart, *Temporal Logic*, Springer-Verlag (N.Y.: 1971).

81. András Brody, *Proportions, Prices, and Planning*, American Elsevier (N.Y.: 1970), p. 84.

82. *Ibid.*, pp. 24-5; 30.

82.5 Charles Musès, "Hypernumbers and Quantum Field Theory", *op. cit.*, p. 81.

83. "We have, therefore, arrived at the classical self-referential situation characterizing paradoxes. Here money is both a commodity and a set of commodities." -- Z. A. Melzak, Mathematical Ideas, Modeling and Applications, Wiley Interscience (N.Y.: 1976), p. 370.

84. Y. Bar-Hillel and A. Fraenkel, Foundations of Set Theory, North Holland (Amsterdam: 1958), p. 41: "A class is called impredicative if it is defined, or definable only, by reference to a totality to which the class itself belongs, and a definition of such kind is impredicative. One may say that a definition written in symbols is impredicative if it defines an object which is one of the values of a bound variable occurring in the defining expression."

85. Raymond L. Wilder, The Foundations of Mathematics, John Wiley (N.Y.: 1952), p. 104. cf. Bertrand Russell, "The Philosophy of Logical Atomism", The Monist 29 (1919), pp. 354; 358.

86. "...a set of all sets or other sets of similar extension cannot exist, since every set obtained in this way immediately gives rise to further applications of the operation 'set of' and, therefore, to the existence of larger sets." Kurt Gödel, "What is Cantor's Continuum Problem?" in Paul Benacerraf and Hillary Putnam, eds., Philosophy of Mathematics: Selected Readings, Prentice-Hall (Englewood Cliffs, N.J.: 1964), p. 263n. See also p. 262n: "The operation 'set of x's' (where the variable 'x' ranges over some given kind of objects) cannot be defined satisfactorily (at least not in the present state of knowledge), but can only be paraphrased by other expressions involving again the concept of set..." (note impredicativity of such "paraphrasing" definitions).

87. W. Fowler, "The Origin of the Elements", Proc. U. S. Nat'l Acad. Sci. 52: 1964, pp. 530-532.

88. Melvin Calvin, Chemical Evolution: Molecular Evolution Towards The Origin of Living Systems on Earth and Elsewhere, Oxford U. Press (N.Y.: 1969), pp. 122-161.

89. A "Heterotroph" is an organism which depends entirely upon other-than-internal sources for its food (in contrast to an "autotroph" or "self-feeder"--e.g. a plant). See A. Oparin, *The Origin of Life*, Dover (N.Y.: 1938), p. 203. "Heterotrophic" in this context refers to organisms that nourish themselves via *fermentative* metabolism, but which lack both the (later to evolve) *photosynthetic* and the *respirative* dimensions of metabolism.

90. The term "cosmogenesis" was coined by T. Chardin to designate the on-going self-creation or evolutionary unfolding of the cosmos grasped as a single or "whole" process, as in a Unitary Field Theory approach. See Pierre Teilhard de Chardin, *The Phenomenon of Man*, Harper & Row (N.Y.: 1959), *passim*.

91. Charles Musès, *The First Nondistributive Algebra*, *op*. *cit*., pp. 209-211.

92. George Boole, *The Mathematical Analysis of Logic*, pp. 15-16: "Let us employ the letters X, Y, Z, to represent the individual members of classes, X applying to every member of one class, as members of that particular class, and Y to every member of another class as members of such class...The symbol x *operating upon* any subject comprehending individuals or classes, shall be supposed to *select* from that subject all the X's which it contains...the symbol y, operating upon any subject, shall be supposed to *select* from it all individuals of the class Y which are comprised in it, and so on. When no subject is expressed, we shall suppose 1 (the Universe) to be the subject understood, so that we shall have $x = x(1)$, the meaning of either term being the *selection* from the Universe of all the X's which it contains and the result of the *operation* being, in common language, the class X, i.e. the class of which each member is an X ... the product xy will represent, in succession, the selection of the class Y, and the selection from the class Y of such individuals of class X as are contained in it, the result being the class whose members are *both* X's and Y's...from the nature of the *operation* which the symbols x, y, z are conceived to represent, we shall designate them as *elective* symbols...It will not be necessary that we should here enter into the analysis of that *mental operation* which we have represented by the elective symbol...but it may probably be referred to an exercise of the faculties of Comparison and Attention (emphasis added)."

93. Benjamin Lee Whorf, *loc*. *cit*.

94. P.C.W. Davies, *The Physics of Time Asymmetry*, University of California Press (Berkeley: 1974), pp. 123; 138; 125.

95. Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, *Gravitation*, W. H. Freeman (San Francisco: 1973), p. 474.

96. *International Dictionary of Applied Mathematics*, Van Nostrand (N.Y.: 1960), p. 825.

97. V. N. Tystovich, *Theory of Turbulent Plasma*, transl. by D. L. Burdick, Consultants' Bureau (N.Y.: 1977), p. 40.

98. F. Tappert and R. Hardin, Bell Laboratories Computer Simulation Film, 1971.

99. Albert Einstein and Leopold Infeld, *The Evolution of Physics*, Simon and Schuster (N.Y.: 1938), pp. 241-242.

100. Louis de Broglie, *Nonlinear Wave Mechanics: A Causal Interpretation*, Elsevier (Amsterdam: 1960), pp. 287-291.

101. U. Enz, "A New Type of Soliton with Particle Properties", *Journal of Mathematical Physics 18*:3(March 1977), pp. 347; 352-353. cf. Charles Musès, "Systemic Stability and Cybernetic Control", *op. cit.*, pp. 161-163: "Another important property of non-linear waves it that...Such waves possess corpuscle-like as well as wave-like properties, and we must look to them for the solution of the wave-particle impasse in modern physics...anything which behaves like both a corpuscle and a wave--as do both light waves and fundamental particles--must be associated with a deep-seated nonlinearity." See also: Charles Musès, "Aspects of Some Crucial Problems in Biological and Medical Cybernetics", *op. cit.*, pp. 252-3.

102. Georg Cantor, *Foundations of a General Theory of Manifolds*: A Mathematical-Philosophical Study in the Theory of the Infinite, *Mathematische Annalen* (Leipzig, 1883). Author's Note to Section 1: "With this word (Manifold, Mannigfaltigkeit) I designate a doctrinal concept (Lehrbegriff) which encompasses a great deal and which so far I have attempted to develop only in the specific form of an arithmetical or geometrical theory of aggregates. By a "manifold" or "aggregate" I generally understand every multiplicity which can be thought of as one, i.e. any totality of definite elements which by means of a law can be bound up into a whole, and I believe that in this I am defining something which is related to the Platonic ειδος (eidos) or ιδεα (idea)...Plato himself indicates that these concepts are of Pythagorean origin..."

103. Ordinarily, the "predicates" of firms, organized as the "base space" coordinate system, are dynamical. That is, the firms change their predicates, and hence their classifications--their *positions* in the geometrized "taxonomy", or base-space--through time, as they evolve. Our "PDE" space will thus become, given such "unstable classifications", an "ODE" space, our PD manifold a manifold of *state-space* trajectories. Our "predicates of firms" become "state of the firm" state-variables.

Thus, with a *non-geographic* base-space, the distinction between PD and OD manifolds appears to dissolve. In fact, even geographic base-spases are state-spaces of a sort, given the state-significant geographical movements -- the changes of address -- exhibited by firms.

However, the state-space arising in these cases is of a special sort. The system being modeled here is the whole economy. The firms are (among) its subsystems or parts. The erstwhile PDE space or taxonomy space is *a state-space of the parts* of the whole system being modeled. The *multiple trajectories* of the state-space record or predict the evolution of the *subsystems*. "Taxonomy Space" is not the state-space of the whole system (here the economy). It does not record a single trajectory describing the evolution of the economy *as a whole* in terms of state-variable measurements meaningful or defined *only* on the economy as a whole (≡"*holistic*" state-variables). It *is* a multi-trajectory space, the *economy's* evolution being described by many different trajectories *at once*. Each starting point of each trajectory signifies a different *part of* the whole system, here an individual *firm*, not a different *initial* condition of the whole economy. It is "firm state-space", not "economy state-space". In a nonlinear system, the trajectories of the parts bear no *simple* (e.g. "additive") relation to the state-space trajectory of their *whole* depicted in terms of its own holistic (irreducible) state-variables.

104. Karl Wittfögel, *Oriental Despotism*, Yale U. Press (New Haven: 1967), pp. 11-48.

105. Morris W. Hirsch and Stephen Smale, *op*. *cit*., pp. 227-8.

106. Conrad H. Waddington, "Concluding Remarks" in *Evolution and Consciousness*, Erich Jantsch and Conrad H. Waddington, eds., Addison-Wesley (Don Mills, Ontario: 1976), pp. 244-247.

107. Morris W. Hirsch and Stephen Smale, *op*. *cit*., pp. 319-321.

108. It is a little known fact that Bernhard Riemann actively pursued a research programme closely resembling the "Unitary Field Theory" of Albert Einstein up until his early death in 1866. The connection between General Relativity and Riemannian Manifold Theory (Non-Euclidean "intrinsic geometry") thus takes on still greater significance. See: "Fragments of a Philosophical Contents" in *The Collected Works Bernhard Riemann*, Henrich Weber and Richard Dedekind, eds., Dover (N.Y.: 1953).

109. Pierre Teilhard de Chardin, *The Phenomenon of Man*, *op*. *cit*., pp. 60-61; 287. Chardin's evolutionary "*Law of Complexity-Consciousness*" describes "psychism" or "consciousness", phenomenologically, as an *effect* of the organization of matter. Psychism is directly *proportional to* the "complexity" of the material matrix manifesting that psychism: the more organization, the more consciousness observable in the behavior.

The "Chardin-complexity" of a body involves not only (i) the number of its elements or parts, but(ii) the number of "*interconnections*" among those parts as well. Thus "Chardin-complexity" *provides a rudimentary* measure of "negentropy" in our sense (provided the concept of "interconnections" is defined in a suitably rigorous, general way).

Chardin further holds that the rising value of this complexity in evolution effects a "convergent curvature" in the space time continuum, in terms of the way in which that curvature defines the laws of evolution. This "bending" of space-time mediates a self-acceleration of evolution into ever higher levels of complexity-consciousness. Thus the idea of generalizing from "mass-energy" to "negentropy" as the quality of substance which determines the bending of the local geometry of the state/space-time manifold, is already emergent in Chardin.

110. Retrodictive testing of global socio-economic (and other) models is already a widespread practice. This involves running a model whose primary purpose is future-predictive, on the data of past periods for which the outcome-values are already known, as a way of assessing the probable accuracy of the model's future forecasts. But models whose main aim is retrodictive/reconstructive have also begun to appear, among the "historical" sciences. See, for example, reports of the CLIMAP Project, such as: W. Lawrence Gates, "Modeling Ice-Age Climate", Science 191, March 18, 1976, pp. 1138-1144.

111. William E. Boyce and Richard C. Di Prima, Elementary Differential Equations and Boundary Value Problems, John Wiley & Sons (Santa Barbara: 1977), pp. 433-4.

112. Ranabir Dütt and P. K. Ghosh, "Nonlinear Correction to Lotka-Volterra Oscillations", Math. Biosciences, 27(1975), p. 15.

113. Kenzi Ishida and Shigero Matsumoto, "Nonequilibrium Thermodynamics of Temporally Oscillating Chemical Reactions", Journal of Theoretical Biology 52 (1975), pp. 343-363.

114. Dr. Robert L. Forward, Analog Science Fiction/Science Fact, 95:8 August 1975), p. 148.

115. D. W. Sciama, The Physical Foundations of General Relativity, Doubleday (N.Y.: 1969), p. 62.